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INTEGRABILITY OF TWO DIMENSIONAL QUASI-HOMOGENEOUS POLYNOMIAL DIFFERENTIAL SYSTEMS

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ABSTRACT. In terms of the conservative-dissipative decomposition of a vector field, we characterize the two dimensional quasi-homogeneous polynomial differential systems with a polynomial first integral (in these systems, polynomial integrability and analytic integrability are equivalent). We also provide an easy method to allow us to compute them and their centers. Finally, as an application, we find the quasi-homogeneous polynomial systems of degree two which have a polynomial first integral (the cubic homogeneous systems, among others, are included).

1. Introduction. In this paper, we deal with polynomial differential systems

$$(\dot{x}, \dot{y})^T = \mathbf{F}_r = (P, Q)^T, \tag{1}$$

where $\mathbf{F}_r$ is a quasi-homogeneous polynomial vector field of degree $r \in \mathbf{N} \cup \{0\}$ with respect to type $\mathbf{t} = (t_1, t_2) \in \mathbf{N}^2$, i.e., for any arbitrary positive real ε, $P(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^{r+t_1}P(x, y)$, $Q(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^{r+t_2}Q(x, y)$. In the particular case that $\mathbf{t} = (1, 1)$, system (1) is a homogeneous polynomial differential system of degree $r + 1$.

We recall that a function H is a first integral of (1) in an open subset U of $\mathbf{R}^2$ if H is a non-constant function in U which is constant on each solution curve of (1). Clearly, if $H \in \mathcal{C}^1(U)$ verifies $\nabla H \cdot \mathbf{F}_r \equiv 0$. If there exists a polynomial (analytic) first integral of (1), it says that it is polynomially (analytically) integrable.

We are interested in analyzing when system (1) is analytically integrable. For systems (1), it is easy to prove that the analytic integrability

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and polynomial integrability are equivalent, since H is an analytic first integral of (1), with $H = H_m + H_{m+1} + \cdots$, its expansion into quasi-homogeneous polynomials of degree $m+i$ with respect to a fixed type $\mathbf{t}$, if and only if each quasi-homogeneous part H_{m+i} is a first integral of system (1) for all i. Therefore, we are interested in studying when system (1) has a quasi-homogeneous polynomial first integral.

Among other applications, the existence of an analytic first integral can be used for determining the local phase portrait at an isolated singular point; in particular, for characterizing when a monodromic singular point (the orbits of the system close to the isolated singular point revolve around it) is a center or a focus. So, it will be interesting to know when a monodromic point is analytically integrable, since in such a case it is a center. However, there are centers which do not have an analytic first integral. An example can be seen in [**6**, page 122].

Other papers related to the problem of integrability and integrability of centers are [**1, 3**].

The results obtained in this paper are closely linked to the conservative-dissipative decomposition of (1), see Lemma 3. In the third section, and in relation to polynomial integrability, Theorems 3.1 and 3.2 offer an easy characterization of the quasi-homogeneous polynomial systems with a polynomial first integral. And, concerning the center problem, Theorem 3.3 finds the centers of system (1).

As an application, the last section shows the quasi-homogeneous polynomial systems of degree two having a polynomial first integral (Theorem 4.1). As far as we know, only degree one polynomially integrable systems have been calculated, see Tsygvintsev [**8**] and Llibre and Zhang [**5**]. Recently, Cairó and Llibre [**2**] found the degree two polynomially integrable systems. Finally, we characterize the centers of these systems (Theorems 4.2 and 4.3), by obtaining centers which are not analytically integrable.

2. Preliminaries. We recall that a function of two variables f is a quasi-homogeneous function of degree $k \geq 0$ with respect to type $\mathbf{t} = (t_1, t_2)$ if $f(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^k f(x,y)$. We will denote $\mathcal{P}_k^{\mathbf{t}}$ the vector space of quasi-homogeneous polynomials of degree k with respect to type $\mathbf{t}$. A two-dimensional vector field $\mathbf{F} = (P,Q)^T$ is quasi-homogeneous of degree k with respect to type $\mathbf{t}$ if $P \in \mathcal{P}_{k+t_1}^{\mathbf{t}}$ and

$Q \in \mathcal{P}^{\mathbf{t}}_{k+t_2}$. The vector space of quasi-homogeneous polynomial vector fields of degree k with respect to type $\mathbf{t}$ will be denoted by $\mathcal{Q}^{\mathbf{t}}_k$.

There is no loss of generality in assuming that t_1 and t_2 are coprime (this can be achieved by canceling common factors) and $t_1 \leq t_2$ (otherwise, we interchange x and y).

The following lemma provides a basis for the vector space $\mathcal{P}^{\mathbf{t}}_k$.

Lemma 1. *If there exist k_1, k_2, k_3 integer numbers with $0 \leq k_1 < t_2$, $0 \leq k_2 < t_1$, $k_3 \geq 0$ with $k = k_1t_1 + k_2t_2 + k_3t_1t_2$, then the vector space $\mathcal{P}^{\mathbf{t}}_k$ is $\mathcal{P}^{\mathbf{t}}_k = \operatorname{span}\{x^{k_1+t_2(k_3-j)}y^{k_2+t_1j}\}_{j=0}^{k_3}$. Otherwise, $\mathcal{P}^{\mathbf{t}}_k = \{0\}$.*

Throughout this paper, given the vector fields $\mathbf{F} = (F_1, F_2)^T$, $\mathbf{G} = (G_1, G_2)^T$, the Lie bracket of both vector fields is defined by $[\mathbf{F}, \mathbf{G}] = D\mathbf{F}.\mathbf{G} - D\mathbf{G}.\mathbf{F}$ where $D\mathbf{F}$ and $D\mathbf{G}$ are the derivatives of $\mathbf{F}$ and $\mathbf{G}$, and the wedge product of two vector fields by $\mathbf{F} \wedge \mathbf{G} = F_1G_2 - F_2G_1$. We denote by $\mathbf{X}_h$ the hamiltonian system associated to h, i.e., $\mathbf{X}_h = (-(\partial h/\partial y), (\partial h/\partial x))^T$, denote $\mathbf{D}_0 = (t_1x, t_2y)^T$ (the dissipative vector field of degree 0 with respect to type $\mathbf{t}$) and we also denote $|\mathbf{t}| = t_1 + t_2$.

Next, we cite some properties of the quasi-homogeneous polynomials and vector fields which are easily obtained.

Lemma 2. *The following properties hold*:

1. *If $U \in \mathcal{P}^{\mathbf{t}}_i$ and $\mathbf{F} \in \mathcal{Q}^{\mathbf{t}}_j$, then $\nabla U \cdot \mathbf{F} \in \mathcal{P}^{\mathbf{t}}_{i+j}$ and $\operatorname{div}(\mathbf{F}) \in \mathcal{P}^{\mathbf{t}}_j$.*
2. *If $U \in \mathcal{P}^{\mathbf{t}}_{i+|\mathbf{t}|}$, then $\mathbf{X}_U \in \mathcal{Q}^{\mathbf{t}}_i$.*
3. *If $\mathbf{F} \in \mathcal{Q}^{\mathbf{t}}_i$ and $\mathbf{G} \in \mathcal{Q}^{\mathbf{t}}_j$, then $\mathbf{F} \wedge \mathbf{G} \in \mathcal{P}^{\mathbf{t}}_{i+j+|\mathbf{t}|}$.*
4. *If $U \in \mathcal{P}^{\mathbf{t}}_i$, then $\nabla U \cdot \mathbf{D}_0 = \mathbf{D}_0 \wedge \mathbf{X}_U = iU$ (Euler theorem for quasi-homogeneous functions).*
5. *If $\mathbf{F} \in \mathcal{Q}^{\mathbf{t}}_i$ then $[\mathbf{F}, \mathbf{D}_0] = i\mathbf{F}$.*

We now prove a result that provides a decomposition of a quasi-homogeneous vector field as a sum of two quasi-homogeneous fields, one conservative (having zero-divergence) and the other dissipative.

Lemma 3. *Every* $\mathbf{F} \in \mathcal{Q}_k^{\mathbf{t}}$ *can be expressed as*

$$\mathbf{F} = \frac{1}{k+|\mathbf{t}|}[\mathbf{X}_{\mathbf{D}_0 \wedge \mathbf{F}} + \operatorname{div}(\mathbf{F})\mathbf{D}_0]. \tag{2}$$

Furthermore, such a decomposition is unique.

Proof. Let $\mathbf{F} = (P, Q)^T \in \mathcal{Q}_k^{\mathbf{t}}$. It is straightforward to show that

$$-\frac{\partial \mathbf{D}_0 \wedge \mathbf{F}}{\partial y} + t_1 x \operatorname{div}(\mathbf{F}) = \left(t_1 x \frac{\partial P}{\partial x} + t_2 y \frac{\partial P}{\partial y}\right) + t_2 P,$$
$$\frac{\partial \mathbf{D}_0 \wedge \mathbf{F}}{\partial x} + t_2 y \operatorname{div}(\mathbf{F}) = \left(t_1 x \frac{\partial Q}{\partial x} + t_2 y \frac{\partial Q}{\partial y}\right) + t_1 Q.$$

As $P \in \mathcal{P}_{k+t_1}^{\mathbf{t}}$ and $Q \in \mathcal{P}_{k+t_2}^{\mathbf{t}}$, from the Euler theorem for quasi-homogeneous polynomial.

We prove the second part. For any $h \in \mathcal{P}_{k+t_1+t_2}^{\mathbf{t}}$, $\mu \in \mathcal{P}_k^{\mathbf{t}}$, it has that

$$\operatorname{div}(\mathbf{X}_h) = 0,$$
$$\operatorname{div}(\mu \mathbf{D}_0) = \nabla \mu \cdot \mathbf{D}_0 + (t_1 + t_2)\mu = (k + t_1 + t_2)\mu.$$

Therefore, if h, μ verify (2), it has that

$$\operatorname{div}(\mathbf{F}) = \frac{1}{k+t_1+t_2}(\operatorname{div}(\mathbf{X}_h) + \operatorname{div}(\mu \mathbf{D}_0)) = \mu,$$
$$\mathbf{D}_0 \wedge \mathbf{F} = \frac{1}{k+t_1+t_2}\mathbf{D}_0 \wedge \mathbf{X}_h = \frac{1}{k+t_1+t_2}\nabla h \cdot \mathbf{D}_0 = h,$$

the result follows. □

Our purpose is to know when the system (1) has a quasi-homogeneous polynomial or analytic first integral at origin. We emphasize that system (1) always has a first integral, since by Lemma 2 and from [**7**], $\mathbf{D}_0 \wedge \mathbf{F}_r = t_1 x Q(x,y) - t_2 y P(x,y)$ is an inverse integrating factor of (1) and therefore $H(x,y) = \int P(x,y)(t_1 x Q(x,y) - t_2 y P(x,y))^{-1}\, dy + f(x)$, satisfying $(\partial/\partial x)H = -Q(t_1 x Q(x,y) - t_2 y P(x,y))^{-1}$, is a first integral of (1). But in general H is not defined at origin, therefore it is neither an analytic first integral at origin, nor a formal first integral.

Before presenting our main results, we make the following considerations:

If $PQ \equiv 0$, then x or y are first integrals of system (1); so system (1) is polynomially integrable.

If P, Q are not coprime, it has that $P = fP'$, $Q = fQ'$ where $\mathbf{F}_{r'} = (P', Q')^T$ is a quasi-homogeneous polynomial vector field of degree $r' < r$ with respect to $\mathbf{t}$. It is easy to prove that H is a first integral of (1) if and only if H is a first integral of $(\dot{x}, \dot{y})^T = \mathbf{F}_{r'}$. Therefore, it is sufficient to study the integrability of the second system.

Also, under the above conditions, we can assume that $\mathbf{D}_0 \wedge \mathbf{F}_r \not\equiv 0$, since otherwise, the vector field $\mathbf{F}_r$ is $1/(r + |\mathbf{t}|)\,\mathrm{div}\,(\mathbf{F}_r)\mathbf{D}_0$ which doesn't have an analytic first integral, since it is radial.

Consequently, we will assume from now on that $\mathbf{D}_0 \wedge \mathbf{F}_r \not\equiv 0$ and P, Q coprime with $PQ \not\equiv 0$.

The following result links the existence of a quasi-homogeneous polynomial first integral of system (1) to the quasi-homogeneous polynomials $\mathrm{div}\,(\mathbf{F}_r)$ and $\mathbf{D}_0 \wedge \mathbf{F}_r$.

Theorem 3.1. *Let system* (1) *be with* $\mathbf{D}_0 \wedge \mathbf{F}_r \not\equiv 0$*,* P, Q *coprime and* $PQ \not\equiv 0$*. System* (1) *has got a polynomial first integral if and only if* $\mathrm{div}\,(\mathbf{F}_r) \equiv 0$*, or else* $\mathbf{D}_0 \wedge \mathbf{F}_r = \prod_{j=1}^k f_j$ *where* $f_1, \ldots, f_k$ *are quasi-homogeneous polynomial of degree* $s_1, \ldots, s_k$ *with respect to type* $\mathbf{t}$*, irreducible in* $\mathbf{K}[x, y]$ (*where* $\mathbf{K}$ *is either* $\mathbf{R}$ *or* $\mathbf{C}$), $k \geq 2$ *and there exist* k *integer numbers, non-negative,* $n_1, n_2, \ldots, n_k$*, not all zero, such that*

(3)
$$\left(\sum_{j=1}^{k}(n_j+1)s_j\right)\mathrm{div}\,(\mathbf{F}_r) = \left(\prod_{j=1}^{k} f_j\right)\sum_{j=1}^{k}\sum_{l=j+1}^{k}(n_l - n_j)\frac{1}{f_j f_l}\nabla f_j \cdot \mathbf{X}_{f_l}.$$

Moreover, in such a case, $\prod_{j=1}^k f_j^{n_j+1}$ *is a quasi-homogeneous polynomial first integral of* (1).

Proof. If $\mathrm{div}\,(\mathbf{F}_r) \equiv 0$, the system is Hamiltonian and $\mathbf{D}_0 \wedge \mathbf{F}_r$ is a polynomial first integral. We henceforth assume that $\mathrm{div}\,(\mathbf{F}_r) \not\equiv 0$.

Necessity. Let $U \in \mathcal{P}_i^{\mathbf{t}}$ be a polynomial first integral of $\mathbf{F}_r = (P, Q)^T$, that is, $\nabla U \cdot \mathbf{F}_r = 0$. As the components of $\mathbf{F}_r$ don't have common factors, it has that $\nabla U = f(-Q, P)$. Thus, the first integral U verifies

$$\mathbf{X}_U = f(P,Q)^T = f\mathbf{F}_r,\ f \in \mathcal{P}^{\mathbf{t}}_{i-r-|\mathbf{t}|}, \quad i \geq r + |\mathbf{t}|. \tag{4}$$

Hence, system (1) has a polynomial first integral if and only if there exists an $f \in \mathcal{P}^{\mathbf{t}}_{i-r-|\mathbf{t}|}$ such that $\mathrm{div}\,(f\mathbf{F}_r) = 0$. Applying Lemma 3,

$$\begin{aligned} 0 = \mathrm{div}\,(f\mathbf{F}_r) &= \nabla f \cdot \mathbf{F}_r + f\,\mathrm{div}\,(\mathbf{F}_r) \\ &= \frac{1}{r+|\mathbf{t}|}\nabla f \cdot \mathbf{X}_{\mathbf{D}_0 \wedge \mathbf{F}_r} \\ &\quad + \frac{i-r-|\mathbf{t}|}{r+|\mathbf{t}|}\,\mathrm{div}\,(\mathbf{F}_r) f + \mathrm{div}\,(\mathbf{F}_r) f. \end{aligned}$$

Thus, we have that

$$\nabla f \cdot \mathbf{X}_{\mathbf{D}_0 \wedge \mathbf{F}_r} = -i\,\mathrm{div}\,(\mathbf{F}_r) f. \tag{5}$$

From (5), $f(x,y) = 0$ is a polynomial invariant curve of the hamiltonian vector field $\mathbf{X}_{\mathbf{D}_0 \wedge \mathbf{F}_r}$. If f is a product of polynomials $g_1, \ldots, g_m$, then every $g_i(x,y) = 0$ is an invariant curve of $\mathbf{X}_{\mathbf{D}_0 \wedge \mathbf{F}_r}$, thus $g_i(x,y) = 0$ is a solution curve which passes through the origin. On the other hand, we know that the solution curves which cross the origin verify $(\mathbf{D}_0 \wedge \mathbf{F}_r)(x,y) = 0$. Thus, it follows that any irreducible factor of f is a factor of $\mathbf{D}_0 \wedge \mathbf{F}_r$. Therefore, if $f_1, \ldots, f_k$ are the irreducible quasi-homogeneous factors of $\mathbf{D}_0 \wedge \mathbf{F}_r$ on $\mathbf{K}[x,y]$, f has the expression $f = \prod_{j=1}^k f_j^{n_j}$ with $\sum_{j=1}^k s_j n_j = i - r - |\mathbf{t}|$, with $n_j \geq 0$.

Thus, the left side of (5) gets

$$\nabla f \cdot \mathbf{X}_{\mathbf{D}_0 \wedge \mathbf{F}_r} = \sum_{j=1}^k n_j \frac{\prod_{j=1}^k f_j^{n_j}}{f_j} \nabla f_j \cdot \mathbf{X}_{\mathbf{D}_0 \wedge \mathbf{F}_r} = f \sum_{j=1}^k n_j \frac{\nabla f_j \cdot \mathbf{X}_{\mathbf{D}_0 \wedge \mathbf{F}_r}}{f_j}.$$

Replacing this in (5) yields

$$\left(r + |\mathbf{t}| + \sum_{j=1}^k n_j s_j \right) \mathrm{div}\,(\mathbf{F}_r) = -\sum_{j=1}^k n_j \frac{1}{f_j} \nabla f_j \cdot \mathbf{X}_{\mathbf{D}_0 \wedge \mathbf{F}_r}. \tag{6}$$

We now prove that $\mathbf{D}_0 \wedge \mathbf{F}_r$ has the prescribed form. First, we prove that $\mathbf{D}_0 \wedge \mathbf{F}_r$ has at least two simple and irreducible factors in $\mathbf{K}[x,y]$.

On the one hand, if $\mathbf{D}_0 \wedge \mathbf{F}_r$ had a unique irreducible factor, that is, $\mathbf{D}_0 \wedge \mathbf{F}_r = f_1^m$ with $m \geq 1$, by imposing (6), it would arrive at $\operatorname{div}(\mathbf{F}_r) \equiv 0$. Thus, the assumption leads us to a contradiction.

On the other hand, if $\mathbf{D}_0 \wedge \mathbf{F}_r = \prod_{l=1}^k f_l^{m_l}$, with some $m_j > 1$, $1 \leq j \leq k$, it would have that

$$\mathbf{X}_{\mathbf{D}_0 \wedge \mathbf{F}_r} = \sum_{l=1}^k m_l \frac{\mathbf{D}_0 \wedge \mathbf{F}_r}{f_l} \mathbf{X}_{f_l} = \left(\prod_{l=1}^k f_l^{m_l - 1} \right) \sum_{l=1}^k m_l \frac{\prod_{l=1}^k f_l}{f_l} \mathbf{X}_{f_l}.$$

So, by (6), $f_j^{m_j-1}$ would be a factor of both, $\mathbf{X}_{\mathbf{D}_0 \wedge \mathbf{F}_r}$ and $\operatorname{div}(\mathbf{F}_r)$, and from Lemma 3, $f_j^{m_j-1}$ would be a factor of $\mathbf{F}_r$ which would contradict the fact that the components are coprime.

Therefore $\mathbf{D}_0 \wedge \mathbf{F}_r$ is $\prod_{l=1}^k f_l$ with $k \geq 2$, and in such a case, the right side of (6) is

$$\begin{aligned} -\sum_{j=1}^k n_j \frac{1}{f_j} \nabla f_j \cdot \mathbf{X}_{\mathbf{D}_0 \wedge \mathbf{F}_r} & \\ &= -\sum_{j=1}^k n_j \sum_{\substack{l=1 \\ l \neq j}}^k \frac{\mathbf{D}_0 \wedge \mathbf{F}_r}{f_j f_l} \nabla f_j \cdot \mathbf{X}_{f_l} \\ &= \sum_{j=1}^k \sum_{l=j+1}^k (n_l - n_j) \frac{\mathbf{D}_0 \wedge \mathbf{F}_r}{f_j f_l} \nabla f_j \cdot \mathbf{X}_{f_l} \\ &= \left(\prod_{j=1}^k f_j \right) \sum_{j=1}^k \sum_{l=j+1}^k (n_l - n_j) \frac{1}{f_j f_l} \nabla f_j \cdot \mathbf{X}_{f_l}. \end{aligned}$$

Thus, (3) holds and consequently the necessity follows.

Sufficiency. It is easy to check that $\prod_{j=1}^k f_j^{n_j+1}$ is a polynomial first integral of $\mathbf{F}_r$, where $\mathbf{D}_0 \wedge \mathbf{F}_r = \prod_{j=1}^k f_j$ and n_j given by (3).

From Theorem 3.1 and Lemma 1, we obtain the following result:

Corollary 3.1 (Necessary condition for polynomial integrability). *Let system* (1) *be with* P, Q *coprime,* $PQ \not\equiv 0$ *and* $\operatorname{div}(\mathbf{F}_r) \not\equiv 0$. *If* *system* (1) *has a polynomial first integral, the decomposition of the quasi-homogeneous polynomial* $\mathbf{D}_0 \wedge \mathbf{F}_r$ *over* $\mathbf{C}[x, y]$ *is*

$$(\mathbf{D}_0 \wedge \mathbf{F}_r)(x, y) = cx^{\delta_x} y^{\delta_y} \prod_{i=1}^{m} (y^{t_1} - \lambda_i x^{t_2}), \tag{7}$$

with $r + |\mathbf{t}| = t_1\delta_x + t_2\delta_y + t_1t_2m$, *where* $c \neq 0$, $\delta_x, \delta_y \in \{0, 1\}$, $\delta_x + \delta_y + m \geq 2$ *and* $\lambda_1, \dots, \lambda_m$ *distinct non-zero complex numbers. If some* $\lambda_i \in \mathbf{C} - \mathbf{R}$, *then there exists a* j *such that* $\lambda_j = \overline{\lambda}_i$.

In what follows, we will use the rational function $\eta := [\operatorname{div}(\mathbf{F}_r)/\mathbf{D}_0 \wedge \mathbf{F}_r]$. We now give a result which simplifies the conditions of polynomial integrability of a quasi-homogeneous polynomial system. This provides an effective way for computing the polynomially integrable systems, to be used in the applications.

Theorem 3.2. *Let system* (1) *be with* P, Q *coprime,* $PQ \not\equiv 0$ *and* $\mathbf{D}_0 \wedge \mathbf{F}_r$ *given by* (7)*. The system* (1) *has a polynomial first integral if and only if* $\operatorname{div}(\mathbf{F}_r) \equiv 0$, *or else there exist* n_x, n_y, n_i, $i = 1, \dots, m$ *non-negative integers, not all zero, verifying*

$$\begin{cases} \operatorname{Res}[\eta(x,1), 0] = -(1/t_2) + [(n_x+1)(r+|\mathbf{t}|)]/(t_2M) & \text{if } \delta_x = 1, \\ \operatorname{Res}[\eta(1,y), 0] = (1/t_1) - [(n_y+1)(r+|\mathbf{t}|)]/(t_1M) & \text{if } \delta_y = 1, \\ \operatorname{Res}[\eta(1,y), \lambda_i^{1/t_1}] = (1/t_1) - [(n_i+1)(r+|\mathbf{t}|)]/(t_1M) & i = 1, \dots, m, \end{cases} \tag{8}$$

where $M = t_1(n_x+1)\delta_x + t_2(n_y+1)\delta_y + t_1t_2\sum_{j=1}^{m}(n_j+1)$.

Proof. We first prove the necessity. If $\operatorname{div}(\mathbf{F}_r) \equiv 0$, the system is Hamiltonian and $\mathbf{D}_0 \wedge \mathbf{F}_r$ is a polynomial first integral. We assume that $\operatorname{div}(\mathbf{F}_r) \not\equiv 0$ and that system (1) has a polynomial first integral. So, taking into account (7), by applying (3), it has that $\operatorname{div}(\mathbf{F}_r)$ is equal to

$$\frac{1}{M}[(n_y - n_x)\delta_x\delta_y \nabla(x) \cdot \mathbf{X}_y \frac{\mathbf{D}_0 \wedge \mathbf{F}_r}{x^{\delta_x} y^{\delta_y}} + \sum_{j=1}^{m} (n_j - n_x)\delta_x \nabla(x) \cdot \mathbf{X}_{(y^{t_1} - \lambda_j x^{t_2})} \frac{\mathbf{D}_0 \wedge \mathbf{F}_r}{x^{\delta_x}(y^{t_1} - \lambda_j x^{t_2})} \tag{9}$$

$$+\sum_{j=1}^{m}(n_j-n_y)\delta_y\nabla(y)\cdot\mathbf{X}_{(y^{t_1}-\lambda_j x^{t_2})}\frac{\mathbf{D}_0\wedge\mathbf{F}_r}{y^{\delta_y}(y^{t_1}-\lambda_j x^{t_2})}$$
$$+\sum_{j=1}^{m}\sum_{l=j+1}^{m}(n_l-n_j)\nabla(y^{t_1}-\lambda_j x^{t_2})$$
$$\cdot\mathbf{X}_{(y^{t_1}-\lambda_l x^{t_2})}\frac{\mathbf{D}_0\wedge\mathbf{F}_r}{(y^{t_1}-\lambda_j x^{t_2})(y^{t_1}-\lambda_l x^{t_2})}.$$

Therefore,

$$\eta(x,y)=\frac{1}{M}\Bigg((n_x-n_y)\delta_x\delta_y\frac{1}{x^{\delta_x}y^{\delta_y}}$$
$$+\sum_{j=1}^{m}t_1(n_x-n_j)\delta_x\frac{y^{t_1-1}}{x^{\delta_x}(y^{t_1}-\lambda_j x^{t_2})}$$
$$+\sum_{j=1}^{m}t_2\lambda_j(n_y-n_j)\delta_y\frac{x^{t_2-1}}{y^{\delta_y}(y^{t_1}-\lambda_j x^{t_2})}$$
$$-\sum_{j=1}^{m}\sum_{l=j+1}^{m}t_1t_2(n_j-n_l)(\lambda_j-\lambda_l)$$
$$\times\frac{x^{t_2-1}y^{t_1-1}}{(y^{t_1}-\lambda_j x^{t_2})(y^{t_1}-\lambda_l x^{t_2})}\Bigg).$$

If $\delta_x=1$, we have that

$$\begin{aligned}\operatorname{Res}[\eta(x,1),0]&=\lim_{x\to 0}x\eta(x,1)\\&=\frac{1}{M}[(n_x-n_y)\delta_y+\sum_{j=1}^{m}t_1(n_x-n_j)]\\&=\frac{(n_x+1)(r+|\mathbf{t}|)-M}{t_2M}.\end{aligned}$$

Analogously, if $\delta_y=1$, it holds

$$\begin{aligned}\operatorname{Res}[\eta(1,y),0]&=\frac{1}{M}[(n_x-n_y)\delta_x-\sum_{j=1}^{m}t_2(n_y-n_j)]\\&=-\frac{(n_y+1)(r+|\mathbf{t}|)-M}{t_1M}.\end{aligned}$$

As for each λ_i, $i=1,\dots,m$, it holds that $\lim_{y\to\lambda_i^{1/t_1}}(y-\lambda_i^{1/t_1})/(y^{t_1}-\lambda_i)=(1/t_1\lambda_i^{1-1/t_1})$, then the residue of $\eta(1,y)$ at each one of the t_1 roots of $y=\lambda_i^{1/t_1}$ is the same, and its value is

$$\begin{aligned}
&\operatorname{Res}[\eta(1,y),\lambda_i^{1/t_1}]\\
&=\lim_{y\to\lambda_i^{1/t_1}}(y-\lambda_i^{1/t_1})\eta(1,y)\\
&=\frac{1}{M}\left((n_x-n_i)\delta_x+\frac{t_2}{t_1}(n_y-n_i)\delta_y+\sum_{j=1}^{m}t_2(n_j-n_i)\right)\\
&=\frac{1}{t_1M}\Bigg(t_1[(n_x+1)-(n_i+1)]\delta_x+t_2[(n_y+1)\\
&\qquad\qquad -(n_i+1)]\delta_y+t_1t_2\sum_{j=1}^{m}[(n_j+1)-(n_i+1)]\Bigg)\\
&=\frac{-(n_i+1)(t_1\delta_x+t_2\delta_y+t_1t_2m)+M}{t_1M}\\
&=\frac{-(n_i+1)(r+|\mathbf{t}|)+M}{t_1M}.
\end{aligned}$$

We now prove the sufficiency. Firstly, with $\mathbf{D}_0\wedge\mathbf{F}_r$ of the form (7) fixed, we prove that the conditions (8) determine univocally $\operatorname{div}(\mathbf{F}_r)$. Furthermore, we prove that (8) provides $m-1+\delta_x+\delta_y$ independent conditions on $\operatorname{div}(\mathbf{F}_r)$. In fact, the degree of the quasi-homogeneous polynomial $\operatorname{div}(\mathbf{F}_r)$ with respect to $\mathbf{t}$ is

$$\begin{aligned}
r&=t_1\delta_x+t_2\delta_y+t_1t_2m-t_1-t_2\\
&=t_1t_2(m-2+\delta_x+\delta_y)+t_1(t_2-1)(1-\delta_x)+t_2(t_1-1)(1-\delta_y).
\end{aligned}$$

Thus, by Lemma 1, $\operatorname{div}(\mathbf{F}_r)(x,y)=x^{(t_2-1)(1-\delta_x)}y^{(t_1-1)(1-\delta_y)}\mu^{\text{hom}}(x^{t_2},y^{t_1})$, where μ^{hom} is a homogenous polynomial of degree $m-2+\delta_x+\delta_y$ which has the expression

$$\mu^{\text{hom}}(x,y)=\sum_{j=0}^{m-2+\delta_x+\delta_y}d_jx^{(m-2+\delta_x+\delta_y-j)}y^j. \tag{10}$$

First, we will prove that the conditions (8) are conditions on $\operatorname{div}(\mathbf{F}_r)\times(1,y)$. On the one hand, if λ is a simple root of $(\mathbf{D}_0\wedge\mathbf{F}_r)(1,y)$, it has that

$$\operatorname{div}(\mathbf{F}_r)(1,\lambda) = \operatorname{Res}[\eta(1,y),\lambda]\frac{\partial(\mathbf{D}_0 \wedge \mathbf{F}_r)}{\partial y}(1,\lambda).$$

On the other hand, we prove that

$$\operatorname{Res}[\eta(1,y),\infty] = -\delta_x \operatorname{Res}[\eta(x,1),0], \tag{11}$$

where, by definition, $\operatorname{Res}[\eta(1,y),\infty] = (1/2\pi i)\oint_{\gamma^-}\eta(1,y)\,dy$ with γ^- any negatively oriented closed curve which contains all the poles of $\eta(1,y)$ in its interior.

In fact, if $\delta_x = 0$, for a sufficiently large R it has

$$\begin{aligned}\left|\oint_{\gamma^-}\eta(1,y)\,dy\right| &= \left|\int_0^{2\pi}\eta(1,\mathrm{Re}^{-i\theta})(-iR)e^{-i\theta}\,d\theta\right| \\ &\le \int_0^{2\pi}\left|\eta(1,\mathrm{Re}^{-i\theta})\right| R\,d\theta\end{aligned}$$

The difference between the degree of the polynomials $(\mathbf{D}_0 \wedge \mathbf{F}_r)(1,y)$ and $\operatorname{div}(\mathbf{F}_r)(1,y)$ is greater than or equal to two, therefore,

$$\left|\oint_{\gamma^-}\eta(1,y)\,dy\right| \le \lim_{R\to\infty}\int_0^{2\pi}\left|\eta(1,Re^{-i\theta})\right| R\,d\theta = 0,$$

hence, (11) holds.

If $\delta_x = 1$ the difference of both degrees is greater than or equal to one, hence $\operatorname{Res}[\eta(1,y),\infty] = -\lim_{y\to\infty} y\eta(1,y)$. By (7) and (10), it is easy to show that

$$\operatorname{Res}[\eta(x,1),0] = \lim_{x\to 0} x\eta(x,1) = \frac{d_{m-1+\delta_y}}{c} = \lim_{y\to\infty} y\eta(1,y),$$

thus, (11) holds.

In summary, the conditions (8) are $m+\delta_x+\delta_y$ conditions on $\mu^{\mathrm{hom}}(1,y)$ and by the residues theorem, there are $m-1+\delta_x+\delta_y$ independent conditions. Since the degree of $\mu^{\mathrm{hom}}(1,y)$ is $m-2+\delta_x+\delta_y$, it holds that $\mu^{\mathrm{hom}}(1,y)$ is univocally defined by (8) and, as a consequence, $\operatorname{div}(\mathbf{F}_r)(1,y)$ is also.

Lastly, we prove that $\mathbf{F}_r$ verifies (3). Let $\widetilde{\mathbf{F}}_r = (1/r+|\mathbf{t}|)[\mathbf{X}_{\mathbf{D}_0\wedge\mathbf{F}_r} + \mu\mathbf{D}_0]$, where μ is the quasi-homogeneous polynomial given by the right side of (9). Trivially, $\widetilde{\mathbf{F}}_r$ verifies (3), thus $\widetilde{\mathbf{F}}_r$ has a polynomial first integral. From the necessary condition, μ satisfies the conditions (8). Therefore, $\mu = \operatorname{div}(\mathbf{F}_r)$. Hence, $\mathbf{F}_r$ is polynomially integrable. □

We finish the section by presenting a result concerning the center problem of system (1). That is, the problem of characterizing when a monodromic point is either a center or a focus.

Theorem 3.3. *Let system* (1) *be with* $\operatorname{div}(\mathbf{F}_r) \not\equiv 0$, $\mathbf{D}_0 \wedge \mathbf{F}_r \not\equiv 0$, P, Q *coprime,* $PQ \not\equiv 0$ *and the origin being a monodromic point. The origin of system* (1) *is a center if and only if* $\sum_{j=1}^{s} \operatorname{Im}(\operatorname{Res}[\eta(1,y),\omega_j]) = 0$ *where* $\omega_1, \dots, \omega_s$ *are the complex roots of* $(\mathbf{D}_0 \wedge \mathbf{F}_r)(1,y)$ *with* $\operatorname{Im}(\omega_j) > 0$.

Proof. From Lemma 2, $[\mathbf{F}_r, \mathbf{D}_0] = r\mathbf{F}_r$, thus $\mathbf{D}_0 \wedge \mathbf{F}_r$ is an inverse integrating factor of (1), see [**7**]. Therefore, their factors are invariant curves of (1) (see [**4**]), and as O is a monodromic point, all the factors of $\mathbf{D}_0 \wedge \mathbf{F}_r$ must be complex. Thus, $\mathbf{D}_0 \wedge \mathbf{F}_r = \prod_{j=1}^{k} f_j$ with $f_j = y^{t_1} - \lambda_j x^{t_2}$ where $\lambda_j \in \mathbf{C} - \mathbf{R}$. Let us note that the λ_j cannot be distinct. In [**1**], it is proved that, in such a case, O is a center if and only if $I := \int_{-\infty}^{\infty} \eta(1,y)\,dy = 0$. It is easy to check that the difference between the degrees of the polynomials $(\mathbf{D}_0 \wedge \mathbf{F}_r)(1,y)$ and $\operatorname{div}(\mathbf{F}_r)(1,y)$ is greater than or equal to two; thus, the above integral is $I = 2\pi i \sum_{j=1}^{s} \operatorname{Res}[\eta(1,y),\omega_j]$, where $\omega_1, \dots, \omega_s$ are the roots of $(\mathbf{D}_0 \wedge \mathbf{F}_r)(1,y)$ with $\operatorname{Im}(\omega_j) > 0$, $1 \le j \le s < k$.

Using the fact that $\operatorname{Res}[\eta(1,y),\overline{\omega}_j] = \overline{\operatorname{Res}[\eta(1,y),\omega_j]}$ and from the residue theorem, it holds that

$$0 = \sum_{j=1}^{s} \operatorname{Res}[\eta(1,y),\omega_j] = 2\operatorname{Re}\left(\sum_{j=1}^{s} \operatorname{Res}[\eta(1,y),\omega_j]\right).$$

The proof is concluded. □

4. Applications to the quasi-homogeneous polynomial systems of degree two. In this section, we characterize the quasi-homogeneous polynomial systems of degree two $(\dot{x}, \dot{y})^T = \mathbf{F}_2$ which have an analytic first integral and the systems which have a center at the origin. For simplicity, we will assume that the vector fields $\mathbf{F}_2$ have coprime components, since otherwise their integrability would be equivalent to the integrability of irreducible quasi-homogeneous vector fields of a minor degree.

According to its type, the quasi-homogeneous vector fields of degree two with $t_1 \leq t_2$ (if $t_1 > t_2$, it interchanges x and y) come given by

$$\begin{aligned}
&\mathbf{F}_2^{(t_1,t_2)}, \quad t_1 > 2,\\
&\mathbf{F}_2^{(2,2n+1)} = (a_1x^2, b_1xy)^T, \quad n \geq 1,\\
&\mathbf{F}_2^{(1,n)} = (a_1x^3, b_1x^{n+2} + b_2x^2y)^T, \quad n \geq 4,\\
&\mathbf{F}_2^{(1,3)} = (a_1x^3 + a_2y, b_1x^5 + b_2x^2y)^T,\\
&\mathbf{F}_2^{(1,2)} = (a_1x^3 + a_2xy, b_1x^4 + b_2x^2y + b_3y^2)^T,\\
&\mathbf{F}_2^{(1,1)} = (a_1x^3+a_2x^2y+a_3xy^2+a_4y^3, b_1x^3+b_2x^2y+b_3xy^2+b_4y^3)^T.
\end{aligned}$$

For $t_1 > 2$, from Lemma 1, $\mathcal{P}_2^{\mathbf{t}} = \{0\}$, hence $\operatorname{div}(\mathbf{F}_2^{(t_1,t_2)})$ is zero. Further, if one of the components of the vector field were non-null, for instance P, from Lemma 1, it would be $2+t_1 = k_1t_1+k_2t_2+k_3t_1t_2$, that is, $k_1 = k_3 = 0$ and $k_2 = 1$, i.e., $t_2 = 2 + t_1$. Thus, $\mathbf{D}_0 \wedge \mathbf{F}_2^{(t_1,t_2)} = cy^2$, hence Q is null.

The components of the fields $\mathbf{F}_2^{(2,2n+1)}$ and $\mathbf{F}_2^{(1,n)}$ have common factors, so we do not consider them.

The necessary condition of integrability, given by Corollary 3.1, provides a pre-classification of the polynomially integrable vector field $\mathbf{F}_2$, according to the factors of $\mathbf{D}_0 \wedge \mathbf{F}_2$.

Proposition 4.1. *A system $(\dot{x}, \dot{y})^T = \mathbf{F}_2$ with $\mathbf{F}_2$ quasi-homogeneous vector field of degree two, with coprime components and non-null, is polynomially integrable if it can be transformed by means of a linear change of variables into one of the following systems*:

1. $(\dot{x}, \dot{y})^T = \mathbf{G}_2^{(1,3)}$ *given by*

(12) $$\dot{x} = (d_2 - c_2)x^3 - 2c_1y, \quad \dot{y} = 6c_3x^5 + 3(c_2 + d_2)x^2y,$$

with c_1, c_2, c_3, d_2 real numbers and $\mathbf{G}_2^{(1,3)}$ with coprime components,

2. $(\dot{x}, \dot{y})^T = \mathbf{G}_2^{(1,2)}$ *given by*

(13) $$\begin{aligned}\dot{x} &= (d_2 - c_2)x^3 + (d_1 - 2c_1)xy,\\ \dot{y} &= 5c_3x^4 + (3c_2 + 2d_2)x^2y + (c_1 + 2d_1)y^2,\end{aligned}$$

with c_1, c_2, c_3, d_1, d_2 *real numbers,* $\mathbf{G}_2^{(1,2)}$ *with coprime components,*

3. $(\dot{x}, \dot{y})^T = \mathbf{G}_{2,a}^{(1,1)}$ *given by*

$$\begin{aligned} \dot{x} &= (d_1 - c_1)x^3 + (d_2 - 2c_2)x^2y + (d_3 - 3c_3)xy^2, \\ \dot{y} &= (3c_1 + d_1)x^2y + (d_2 + 2c_2)xy^2 + (d_3 + c_3)y^3, \end{aligned} \tag{14}$$

with $c_1, c_2, c_3, d_1, d_2, d_3$ *real numbers,* $\mathbf{G}_{2,a}^{(1,1)}$ *with coprime components,*

4. $(\dot{x}, \dot{y})^T = \mathbf{G}_{2,b}^{(1,1)}$ *given by*

$$\begin{aligned} \dot{x} &= (d_0 - c_2)x^3 + (d_1 - 2c_1 - 2c_3)x^2y + (d_2 - 3c_2)xy^2 - 4c_3y^3, \\ \dot{y} &= 4c_1x^3 + (3c_2 + d_0)x^2y + (d_1 + 2c_1 + 2c_3)xy^2 + (d_2 + c_2)y^3, \end{aligned} \tag{15}$$

where $c_1, c_2, c_3, d_0, d_1, d_2$ *real numbers,* $\mathbf{G}_{2,b}^{(1,1)}$ *with coprime components and* $c_2^2 - 4c_1c_3 < 0$.

Proof. It is easy to check that the vector fields $\mathbf{G}_2^{(1,3)}$ and $\mathbf{G}_2^{(1,2)}$ are $\mathbf{F}_2^{(1,3)}$ and $\mathbf{F}_2^{(1,2)}$, by rewriting their coefficients, respectively.

From the necessary condition of polynomial integrability, the factors of the decomposition of the polynomial $\mathbf{D}_0 \wedge \mathbf{F}_2^{(1,1)}$ over $\mathbf{R}$ must be simple. Thus, $\mathbf{D}_0 \wedge \mathbf{F}_2^{(1,1)}$ either has at least two distinct real roots or has no real roots. In the first case, $\mathbf{D}_0 \wedge \mathbf{F}_2^{(1,1)} = (\alpha_1 x + \beta_1 y)(\alpha_2 x + \beta_2 y)p_2(x, y)$ where p_2 is a homogeneous polynomial of degree two in x, y. Making $u = \alpha_1 x + \beta_1 y$, $v = \alpha_2 x + \beta_2 y$, the system is transformed into $(\dot{u}, \dot{v})^T = \widetilde{\mathbf{F}}_2^{(1,1)}$ with $\mathbf{D}_0 \wedge \widetilde{\mathbf{F}}_2^{(1,1)} = uv(c_1u^2 + c_2uv + c_3v^2)$. Hence, $\widetilde{\mathbf{F}}_2^{(1,1)}$ is $\mathbf{G}_{2,a}^{(1,1)}$.

In the second case, $\mathbf{D}_0 \wedge \mathbf{F}_2^{(1,1)} = [(y - \alpha x)^2 + \beta^2 x^2]q_2(x, y)$ where $\beta \neq 0$ and q_2 is a homogeneous polynomial of degree two in x, y with imaginary roots. Letting $u = x$, $v = -(\alpha/\beta)x + (1/\beta)y$, the system is transformed into $(\dot{u}, \dot{v})^T = \widetilde{\mathbf{F}}_2^{(1,1)}$ with $\mathbf{D}_0 \wedge \widetilde{\mathbf{F}}_2^{(1,1)} = (v^2 + u^2)(c_1u^2 + c_2uv + c_3v^2)$ with $c_2^2 - 4c_1c_3 < 0$. So, $\widetilde{\mathbf{F}}_2^{(1,1)} = \mathbf{G}_{2,b}^{(1,1)}$. □

Next, we determine the polynomial integrability of the families given in Proposition 4.1 and, as a consequence, we will compute all the irreducible quasi-homogeneous polynomial systems of degree two which are polynomially integrable.

We study each of them separately.

Proposition 4.2. *The system* $(\dot{x}, \dot{y})^T = \mathbf{G}_2^{(1,3)}$ *with* $\mathbf{G}_2^{(1,3)}$ *having coprime components is polynomially integrable if and only if*

i) $d_2 = 0$ (*Hamiltonian system*), *or*

ii) $d_2 \neq 0$, $c_1 \neq 0$, $d_2(n_1 + n_2 + 2) = (n_2 - n_1)\sqrt{\Delta}$ *where* $\Delta := c_2^2 - 4c_1c_3 > 0$, *with* n_1, n_2 *any non-negative integer numbers and where at least one is non-zero.*

Proof. The vector field $\mathbf{G}_2^{(1,3)}$ is quasi homogeneous of degree 2 with respect to type $(1,3)$. Their conservative and dissipative parts are

$$\mathbf{D}_0 \wedge \mathbf{G}_2^{(1,3)} = 6(c_1y^2 + c_2x^3y + c_3x^6), \quad \operatorname{div}(\mathbf{G}_2^{(1,3)}) = 6d_2x^2.$$

Thus, if $d_2 = 0$, system (12) is Hamiltonian, case i). We assume that $d_2 \neq 0$. If the discriminant Δ or c_1 are zero then the polynomial $\mathbf{D}_0 \wedge \mathbf{G}_2^{(1,3)}$ has multiple factors. And if $\Delta < 0$, $\mathbf{D}_0 \wedge \mathbf{G}_2^{(1,3)}$ has only an irreducible factor. In both cases, by Theorem 3.1, system (12) is not polynomially integrable.

If $\Delta > 0$ and $c_1 \neq 0$, it has that $\mathbf{D}_0 \wedge \mathbf{G}_2^{(1,3)} = 6c_1(y^2 - (c_2/c_1)x^3y - (c_3/c_1)x^6) = 6c_1(y - \lambda_1x^3)(y - \lambda_2x^3)$, with λ_1, λ_2 distinct non-zero real numbers. By (8), system (12) is polynomially integrable if and only if there exist n_1, n_2 non-negative integer numbers, not all zero, verifying

$$\operatorname{Res}[\eta(1,y), \lambda_i] = \frac{d_2}{c_1(\lambda_i - \lambda_j)} = -\frac{(n_i + 1)6 - M}{M}, \quad i, j = 1, 2, \ i \neq j$$

where $M = 3n_1 + 3n_2 + 6$. And as $c_1(\lambda_1 - \lambda_2) = \sqrt{\Delta}$, it holds that $d_2(n_1 + n_2 + 2) = (n_2 - n_1)\sqrt{\Delta}$ (case ii)). ◻

Proposition 4.3. *The system* $(\dot{x}, \dot{y})^T = \mathbf{G}_2^{(1,2)}$ *with* $\mathbf{G}_2^{(1,2)}$ *having coprime components is polynomially integrable if and only if one of the following series of conditions holds*:

i) $d_1 = d_2 = 0$ (*Hamiltonian system*),

ii) $d_1 \neq 0$ *or* $d_2 \neq 0$, $\Delta < 0$, $d_1c_2 = 2d_2c_1$, $d_2(n_x + 4n_1 + 5) = (n_x - n_1)c_2$, *where* n_x, n_1 *any non-negative integer numbers and at least one is non-zero,*

iii) $d_1 \neq 0$ *or* $d_2 \neq 0$, $\Delta > 0$, $c_1 \neq 0$ *and*

$$d_1 = \frac{2n_x - n_1 - n_2}{n_x + 2n_1 + 2n_2 + 5} c_1,$$
$$d_2 = \frac{2n_x - n_1 - n_2}{2(n_x + 2n_1 + 2n_2 + 5)} c_2 + \frac{5(n_2 - n_1)}{2(n_x + 2n_1 + 2n_2 + 5)} \sqrt{\Delta},$$

where n_x, n_1, n_2 *any non-negative integer numbers and not all zeros and* $\Delta := c_2^2 - 4c_1c_3$.

Proof. In this case, we have that

$$\mathbf{D}_0 \wedge \mathbf{G}_2^{(1,2)} = 5x(c_1y^2 + c_2x^2y + c_3x^4), \quad \mathrm{div}\,(\mathbf{G}_2^{(1,2)}) = 5(d_1y + d_2x^2).$$

We assume that either $d_1 \neq 0$ or $d_2 \neq 0$; otherwise, system (13) is polynomially integrable since it is Hamiltonian (case i).

If $\Delta < 0$, it has $c_1c_3 \neq 0$ and $\mathbf{D}_0 \wedge \mathbf{G}_2^{(1,2)} = 5c_1x(y - \lambda x^2)(y - \overline{\lambda} x^2)$, with $\lambda \in \mathbf{C} - \mathbf{R}$. By applying Theorem 3.2, with $\delta_x = 1$, $\delta_y = 0$, system (13) will be polynomially integrable if there exist n_x, n_1 ($n_2 = n_1$) non-negative integer numbers and where at least one is non-zero, such that

$$d_1 = \frac{2(n_x - n_1)}{n_x + 4n_1 + 5} c_1, \qquad d_2 = \frac{(n_x - n_1)}{n_x + 4n_1 + 5} c_2, \text{ (case ii).}$$

If $\Delta > 0$ and $c_1 \neq 0$, it has that $\mathbf{D}_0 \wedge \mathbf{G}_2^{(1,2)} = 5c_1x((c_3/c_1)x^4 + (c_2/c_1)x^2y + y^2) = 5c_1x(y - \lambda_1x^2)(y - \lambda_2x^2)$ where $\lambda_1 + \lambda_2 = -(c_2/c_1)$ and $\lambda_1\lambda_2 = (c_3/c_1)$. In this case, system (13) will be polynomially integrable if

$$\mathrm{Res}\,[\eta(x,1), 0] = \frac{d_1}{c_1} = \frac{5(n_x + 1) - M}{2M},$$
$$\mathrm{Res}\,[\eta(1,y), \lambda_i] = \frac{d_2 + d_1\lambda_i}{c_1(\lambda_i - \lambda_j)} = -\frac{5(n_i + 1) - M}{M}, \quad i, j = 1, 2,\ i \neq j,$$

where $M = n_x + 2n_1 + 2n_2 + 5$ with $\lambda_{i,j} = (-c_2 \pm \sqrt{\Delta})/2c_1$. Thus, one arrives at case iii).

If $\Delta = 0$ or $c_1 = 0$, the polynomial $\mathbf{D}_0 \wedge \mathbf{G}_2^{(1,2)}$ has multiple factors and therefore the system does not have a polynomial first integral. And if $\Delta < 0$, $\mathbf{D}_0 \wedge \mathbf{G}_2^{(1,2)}$ has only an irreducible factor. Thus, system (13) is not polynomially integrable. □

Proposition 4.4. *The system* $(\dot{x}, \dot{y})^T = \mathbf{G}_{2,a}^{(1,1)}$ *with* $\mathbf{G}_{2,a}^{(1,1)}$ *having coprime components is polynomially integrable if and only if one of the following series of conditions holds*:

i) $d_1 = d_2 = d_3 = 0$ (*Hamiltonian system*),

ii) d_1, d_2, d_3 *are not zero simultaneously,* $\Delta < 0$, *and*

$$d_3 = \frac{3n_x - n_y - 2n_1}{n_x + n_y + 2n_1 + 4} c_3, \quad d_1 = \frac{n_x - 3n_y + 2n_1}{n_x + n_y + 2n_1 + 4} c_1,$$
$$d_2 = \frac{2(n_x - n_y)}{n_x + n_y + 2n_1 + 4} c_2$$

where n_x, n_y, n_1 *any non-negative integer numbers and at least one is non-zero.*

iii) d_1, d_2 *or* d_3 *are different to zero,* $\Delta > 0$, $c_1 c_3 \neq 0$ *and*

$$d_3 = \frac{3n_x - n_y - n_1 - n_2}{n_x + n_y + n_1 + n_2 + 4} c_3, \quad d_1 = \frac{-n_x + 3n_y - n_1 - n_2}{n_x + n_y + n_1 + n_2 + 4} c_1,$$
$$d_2 = \frac{2(n_x - n_y)}{n_x + n_y + n_1 + n_2 + 4} c_2 - \frac{2(n_1 - n_2)}{n_x + n_y + n_1 + n_2 + 4} \sqrt{\Delta},$$

where n_x, n_y, n_1, n_2 *any non-negative integer numbers and not all zero, being* $\Delta := c_2^2 - 4c_1c_3$.

Proof. The vector field $\mathbf{G}_{2,a}^{(1,1)}$ is a quasi-homogeneous of degree two with respect to type $(1,1)$. We have

$$\mathbf{D}_0 \wedge \mathbf{G}_{2,a}^{(1,1)} = 4xy(c_1x^2 + c_2xy + c_3y^2),$$
$$\operatorname{div}(\mathbf{G}_{2,a}^{(1,1)}) = 4(d_1x^2 + d_2xy + d_3y^2).$$

If $d_1 = d_2 = d_3 = 0$, case i). We assume that $\operatorname{div}(\mathbf{G}_{2,a}^{(1,1)}) \not\equiv 0$. If $\Delta < 0$, one has that $c_1c_3 \neq 0$ and $\mathbf{D}_0 \wedge \mathbf{G}_{2,a}^{(1,1)} = 4c_3xy(y - \lambda x)(y - \overline{\lambda} x)$, with $\lambda \in \mathbf{C} - \mathbf{R}$. From Theorem 3,2, system (14) is polynomially integrable if and only if there exist n_x, n_y, n_1, n_2 such that

$$\begin{aligned}
\text{Res}\,[\eta(x,1),0] &= \frac{d_3}{c_3} = -1 + \frac{4(n_x+1)}{M},\\
\text{Res}\,[\eta(1,y),0] &= \frac{d_1}{c_1} = 1 - \frac{4(n_y+1)}{M},\\
\text{Res}\,[\eta(1,y),\lambda] &= \frac{d_1+d_2\lambda+d_2\lambda^2}{c_3\lambda(\lambda-\overline{\lambda})} = -\frac{4(n_1+1)-M}{M},\\
\text{Res}\,[\eta(1,y),\overline{\lambda}] &= \frac{d_1+d_2\overline{\lambda}+d_3\overline{\lambda}^2}{c_3\overline{\lambda}(\overline{\lambda}-\lambda)} = -\frac{4(n_2+1)-M}{M},
\end{aligned}$$

where $n_2 = n_1$ and $M = n_x + n_y + 2n_1 + 4$. So, one arrives at case ii).

If $\Delta > 0$ and $c_1c_3 \neq 0$, one has that $\mathbf{D}_0 \wedge \mathbf{G}_{2,a}^{(1,1)} = 4c_3xy(y-\lambda_1 x)(y-\lambda_2 x)$ where $\lambda_1+\lambda_2 = -(c_2/c_3)$ and $\lambda_1\lambda_2 = (c_1/c_3)$. Thus, system (14) will be polynomially integrable if

$$\begin{aligned}
\text{Res}\,[\eta(x,1),0] &= \frac{d_3}{c_3} = \frac{4(n_x+1)-M}{M},\\
\text{Res}\,[\eta(1,y),0] &= \frac{d_1}{c_1} = -\frac{4(n_y+1)-M}{M},\\
\text{Res}\,[\eta(1,y),\lambda_i] &= \frac{d_1+d_2\lambda_i+d_3\lambda_i^2}{c_3\lambda_i(\lambda_i-\lambda_j)}\\
&= -\frac{4(n_i+1)-M}{M}, \quad i,j=1,2,\ i\neq j,
\end{aligned}$$

where $M = n_x + n_y + n_1 + n_2 + 4$ with $\lambda_{i,j} = (-c_2 \pm \sqrt{\Delta})/2c_3$. So, one has iii).

In the remaining situations, system (14) is not polynomially integrable. □

Proposition 4.5. *The system* $(\dot{x},\dot{y})^T = \mathbf{G}_{2,b}^{(1,1)}$ *with* $\mathbf{G}_{2,b}^{(1,1)}$ *having coprime components and* $\Delta := c_2^2 - 4c_1c_3 < 0$, *is polynomially integrable if and only if one of the following series of conditions hold*:

i) $d_0 = d_1 = d_2 = 0$ (*Hamiltonian system*),

ii) d_0, d_1, d_2 *are not zero simultaneously and*

$$d_1 = 2\frac{n_3-n_1}{n_1+n_3+2}(c_1-c_3), \qquad d_0 = -d_2 = -\frac{n_3-n_1}{n_1+n_3+2}c_2,$$

where n_1, n_3 any non-negative integer numbers and at least one is non-zero.

Proof. In this case, $\mathbf{D}_0 \wedge \mathbf{G}_{2,b}^{(1,1)} = 4(x^2+y^2)(c_1x^2+c_2xy+c_3y^2)$ and $\operatorname{div}(\mathbf{G}_{2,b}^{(1,1)}) = 4(d_0x^2+d_1xy+d_2y^2)$. If $d_0 = d_1 = d_2 = 0$, system (15) is integrable since it is Hamiltonian (case i)).

We assume that $\operatorname{div}(\mathbf{G}_{2,b}^{(1,1)}) \not\equiv 0$. As $\Delta < 0$, one has that $c_1c_3 \neq 0$ and $\mathbf{D}_0 \wedge \mathbf{G}_{2,b}^{(1,1)} = 4c_3(y+ix)(y-ix)(y-\lambda x)(y-\overline{\lambda}x)$, with $\lambda = (-c_2+\sqrt{-\Delta}i)/2c_3 \in \mathbf{C}-\mathbf{R}$. Let us assume that $c_2^2+(c_1-c_3)^2 \neq 0$, since otherwise $\mathbf{D}_0 \wedge \mathbf{G}_{2,b}^{(1,1)}$ has multiple factors. From Theorem 3.2, system (15) is polynomially integrable if and only if there exist n_1, n_2, n_3, n_4 such that

$$\operatorname{Res}[\eta(1,y), i] = \frac{d_0+d_1i-d_2}{2c_3i(i-\lambda)(i-\overline{\lambda})} = -\frac{4(n_1+1)-M}{M},$$
$$\operatorname{Res}[\eta(1,y), -i] = \frac{d_0-d_1i-d_2}{-2c_3i(-i-\lambda)(-i-\overline{\lambda})} = -\frac{4(n_2+1)-M}{M},$$
$$\operatorname{Res}[\eta(1,y), \lambda] = \frac{d_0+d_1\lambda+d_2\lambda^2}{c_3(\lambda-i)(\lambda+i)(\lambda-\overline{\lambda})} = -\frac{4(n_3+1)-M}{M},$$

$$\operatorname{Res}[\eta(1,y), \bar{\lambda}] = \frac{d_0+d_1\overline{\lambda}+d_2\overline{\lambda}^2}{c_3(\overline{\lambda}-i)(\overline{\lambda}+i)(\overline{\lambda}-\lambda)} = -\frac{4(n_4+1)-M}{M},$$

with $n_2 = n_1$, $n_4 = n_3$ and $M = 2n_1+2n_3+4$. So, it has ii). □

In summary, we have obtained the following result.

Theorem 4.1. *System $(\dot{x},\dot{y})^T = \mathbf{F}_2$ with $\mathbf{F}_2$ quasi-homogeneous vector field of degree two having non-null coprime components, is polynomially integrable if and only if it can be transformed, by means of a linear change of variables, into one of the systems given by the series of conditions of the Propositions* 4.2, 4.3, 4.4 *or* 4.5.

Finally, we characterize the centers of these systems. We assume, therefore, that $\mathbf{D}_0 \wedge \mathbf{F}_2$ has no real factors. Therefore, only systems $(\dot{x},\dot{y})^T = \mathbf{G}_2^{(1,3)}$ and $(\dot{x},\dot{y})^T = \mathbf{G}_{2,b}^{(1,1)}$ can be centers.

Theorem 4.2. *System* (12) *has a center at the origin if and only if* $d_2 = 0$ *and* $c_2^2 - 4c_1c_3 < 0$.

Proof. The origin is monodromic if and only if $\mathbf{D}_0 \wedge \mathbf{G}_2^{(1,3)} = 6(c_1y^2 + c_2x^3y + c_3x^6)$ has no real factors. Thus, we assume that $c_2^2 - 4c_1c_3 < 0$.

If $d_2 = 0$, O is a center, since (12) is monodromic and Hamiltonian. Otherwise, ($d_2 \neq 0$), one has that $\mathbf{D}_0 \wedge \mathbf{G}_2^{(1,3)}(1, y) = 6c_1(y - \lambda)(y - \overline{\lambda})$ with $\lambda \in \mathbf{C} - \mathbf{R}$ non-zero. By Theorem 3.3, system (12) has a center if and only if

$$\operatorname{Im}(\operatorname{Res}[\eta(1, y), \lambda]) = \operatorname{Im}\left(\frac{d_2}{c_1(\lambda - \overline{\lambda})}\right) = -\frac{d_2}{c_1\sqrt{4c_1c_3 - c_2^2}} = 0.$$

This contradicts the fact that $d_2 \neq 0$. □

As a consequence of Proposition 4.2 and Theorem 4.2, we obtain the following result:

Corollary 4.1. *The centers of system* (12) *are Hamiltonian* (*all are integrable centers*).

Theorem 4.3. *System* (15) *has a center at the origin if and only if one of the following series of conditions holds*:

i) $c_1 = c_3 \neq 0$, $c_2 = 0$, $d_0 + d_2 = 0$,

ii) $(c_2^2 + (c_1 - c_3)^2)(d_0 + d_2) = (c_1 + c_3 - \sqrt{\Delta})(d_1c_2 + (d_2 - d_0)(c_3 - c_1))$, *with* $(c_1 - c_3)^2 + c_2^2 \neq 0$ *and* $\Delta := 4c_1c_3 - c_2^2 > 0$.

Proof. For this system, $\mathbf{D}_0 \wedge \mathbf{G}_{2,b}^{(1,1)} = 4(x^2 + y^2)(c_1x^2 + c_2xy + c_3y^2)$. The origin is monodromic if and only if $c_2^2 - 4c_1c_3 < 0$. So, $\mathbf{D}_0 \wedge \mathbf{G}_{2,b}^{(1,1)}(1, y) = 4c_3(y - i)(y + i)(y - \lambda)(y - \overline{\lambda})$ with $\lambda \in \mathbf{C} - \mathbf{R}$.

We distinguish two cases: if $c_2 = c_1 - c_3 = 0$, i.e., i and $-i$ double roots of $\mathbf{D}_0 \wedge \mathbf{G}_{2,b}^{(1,1)}(1, y)$, one has that

$$\begin{aligned}\operatorname{Res}[\eta(1, y), i] &= \lim_{y \to i} \frac{d}{dy}((y - i)^2\eta(1, y)) = \lim_{y \to i} \frac{d}{dy}\left(\frac{d_0 + d_1y + d_2y^2}{c_3(y + i)^2}\right) \\ &= -\frac{(d_0 + d_2)i}{4c_3}.\end{aligned}$$

Therefore, by applying Theorem 3.3, system (15) has a center, in this case, if and only if $d_0 + d_2 = 0$.

And, if $c_2^2 + (c_1 - c_3)^2 > 0$, i.e., the roots of $\mathbf{D}_0 \wedge \mathbf{G}_{2,b}^{(1,1)}(1, y)$ are $i, -i, \lambda$ and $\overline{\lambda}$ with $\lambda \in \mathbf{C}$ and $\mathrm{Re}\,(\lambda) \neq 0$. In such a case, by applying Theorem 3.3, O is a center if and only if the sum of the following expressions is zero

$$\begin{aligned}
\mathrm{Im}\,(\mathrm{Res}\,[\eta(1, y), i]) &= \tfrac{(c_3-c_1)(d_0-d_2)-d_1c_2}{2((c_1-c_3)^2+c_2^2)},\\
\mathrm{Im}\,(\mathrm{Res}\,[\eta(1, y), \lambda]) &= -\tfrac{2(c_3-c_1)(d_0c_3-d_2c_1)-(c_1+c_3)d_1c_2+(d_0+d_2)c_2^2}{2((c_1-c_3)^2+c_2^2)\sqrt{\Delta}}.
\end{aligned}$$

Thus, we arrive at ii). □

Finally, we show the following result, which is easily obtained from Theorem 4.3 and Proposition 4.5.

Corollary 4.2. *We assume that system* (15) *has a center at the origin. Then, the origin is not analytically integrable if and only if one of the following series of conditions holds*:

i) $c_1 = c_3 \neq 0$, $c_2 = 0$, $d_0 \neq 0$,

ii) $c_1 = c_3 \neq 0$, $c_2 = 0$, $d_0 = 0$, $d_1 \neq 0$,

iii) $c_1 \neq c_3$, $c_2 = 0$, $(n_1 + n_3 + 2)d_1 \neq 2(n_3 - n_1)(c_1 - c_3)$, *for any* n_1, n_3 *non-negative integer numbers and where at least one is non-zero,*

iv) $(c_1 - c_3)^2 + c_2^2 \neq 0$, $d_0 + d_2 \neq 0$,

v) $(c_1 - c_3)^2 + c_2^2 \neq 0$, $d_0 + d_2 = 0$, $(n_1 + n_3 + 2)d_2 \neq (n_3 - n_1)c_2$, *for any* n_1, n_3 *non-negative integer numbers and where at least one is non-zero.*

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