

Nilpotent Systems Admitting an Algebraic Inverse Integrating Factor over $\mathbb{C}((x, y))$

Antonio Algaba · Cristóbal García · Manuel Reyes

Received: 19 January 2011 / Accepted: 24 April 2011 / Published online: 8 June 2011
© Springer Basel AG 2011

Abstract We characterize the nilpotent systems whose lowest degree quasi-homogeneous term is $(y, \sigma x^n)^T$, $\sigma = \pm 1$, which have an algebraic inverse integrating factor over $\mathbb{C}((x, y))$. In such cases, we show that the systems admit an inverse integrating factor of the form $(h + \dots)^q$ with $h = 2\sigma x^{n+1} - (n+1)y^2$ and q a rational number. We analyze its uniqueness modulus a multiplicative constant.

1 Introduction and Statement of the Main Results

We consider the system of differential equations given by

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = (P(\mathbf{x}), Q(\mathbf{x}))^T, \quad (1)$$

where \mathbf{F} is a formal planar vector field defined in a neighborhood of the origin $U \subset \mathbb{C}^2$.

A non-null \mathcal{C}^1 class function V is an inverse integrating factor of system (1) (or also of \mathbf{F}) on U if satisfies the linear partial differential equation $L_{\mathbf{F}}V = \text{div}(\mathbf{F})V$, being $L_{\mathbf{F}}V := P\partial V/\partial x + Q\partial V/\partial y$, the Lie derivative of V respect to \mathbf{F} , and $\text{div}(\mathbf{F}) := \partial P/\partial x + \partial Q/\partial y$, the divergence of \mathbf{F} . This name for V comes from the fact that V^{-1} defines on $U \setminus \{V = 0\}$ an integrating factor of system (1), i.e. \mathbf{F}/V is divergence-free. Concretely,

$$H = - \int P/V dy + \int \left(Q/V + \frac{\partial}{\partial x} \int P/V dy \right) dx$$

A. Algaba · C. García · M. Reyes (✉)
Departamento de Matemáticas, Facultad de Ciencias Experimentales,
Universidad de Huelva, Campus del Carmen, Avda. Tres de Marzo s/n,
21071 Huelva, Spain
e-mail: colume@uhu.es

would be a first integral of the system on $U \setminus \{V = 0\}$. We recall that a function H is a first integral of (1) (or also of \mathbf{F}) on U if H is a non-constant function on U which is constant on each solution curve of (1). Clearly, if $H \in \mathcal{C}^1(U)$ verifies $L_{\mathbf{F}}H \equiv 0$. We note that the existence of a first integral of the system (1) determines completely its phase portrait. Among others applications, the existence of an analytic first integral defined in a neighborhood of the origin can be used to characterize when a monodromic singular point (the orbits of the system close to the isolated singular point revolve around it) is a center or a focus, see [4].

There are other reasons to study the existence of inverse integrating factors, among them: this concept plays an important role in the study of the existence of limit cycles of a vector field, because the zero-set $\{V = 0\}$, formed by orbits of the system (1), contains the limit cycles of the system (1) which are in U , whenever they exist, see [10, 14, 15]. The zero-set $\{V = 0\}$ also contains the homoclinic and heteroclinic connections between hyperbolic saddle equilibria, see [13]. Moreover, the cyclicity of a limit cycle is related with the vanishing order of V , see [12]. The expressions of V usually are simpler than the expressions of the first integrals, see [5, 6]. The domain of definition and the regularity of V usually are larger than the domain and the regularity of the first integral, see [7, 11, 16, 18].

The existence of inverse integrating factors in a neighborhood of a singularity has been studied in some particular cases. Concretely, Enciso and Peralta-Salas [10] study the existence of a smooth inverse integrating factor in a neighborhood of an arbitrary elementary singularity, i.e. systems whose linear part at the origin has at least one eigenvalue different from zero. These results extend previous results given in [6], [8, Theorem 5.2], where the authors consider elementary singularities that admit analytic orbital normalization. Algaba et al. [3] characterize the nilpotent systems with a formal inverse integrating factor.

We are interested in characterizing nilpotent systems which have an algebraic inverse integrating factor over $\mathbb{C}((x, y))$ (which will be named AIIF) where $\mathbb{C}((x, y))$ denotes the quotient field of the algebra of the power series $\mathbb{C}[[x, y]]$. The only result that we know in this sense is the following:

Theorem 1 (Walcher [19]) *If the analytic system*

$$(\dot{x}, \dot{y})^T = \left(y, \sum_{j \geq 2} \alpha_j x^j + \beta_j x^{j-1} y \right)^T \quad (2)$$

with $\alpha_2 \neq 0$ and $\beta_2 \neq 0$ (non-degenerate cusp case) has an AIIF, then it is equal to $(y^2 + h.o.t.)^{7/6} \exp(u)$, for some series u which is unique up to an additive constant.

The main goal of this paper is to characterize the nilpotent systems which have an AIIF. As a particular case, we include some necessary conditions for the existence of an AIIF in the non-degenerate cusp case, by extending the above result.

Before showing our results, we recall the following concepts and definitions. Given $\mathbf{t} = (t_1, t_2)$ with t_1 and t_2 natural numbers, a function f of two variables is quasi-homogeneous of type \mathbf{t} and degree k if $f(\varepsilon^{t_1} x, \varepsilon^{t_2} y) = \varepsilon^k f(x, y)$. The vector space of quasi-homogeneous polynomials of type \mathbf{t} and degree k will be denoted by $\mathcal{P}_k^{\mathbf{t}}$.

A vector field $\mathbf{F} = (F_1, F_2)^T$ is quasi-homogeneous of type \mathbf{t} and degree k if $F_1 \in \mathcal{P}_{k+t_1}^{\mathbf{t}}$ and $F_2 \in \mathcal{P}_{k+t_2}^{\mathbf{t}}$. We will denote $\mathcal{Q}_k^{\mathbf{t}}$ the vector space of the quasi-homogeneous polynomial vector fields of type \mathbf{t} and degree k .

Any formal vector field or function can be expanded into quasi-homogeneous terms of type \mathbf{t} of successive degrees. Thus, every vector field \mathbf{F} can be written as

$$\mathbf{F} = \mathbf{F}_r + \mathbf{F}_{r+1} + \dots,$$

for some $r \in \mathbb{Z}$, where $\mathbf{F}_j = (P_{j+t_1}, Q_{j+t_2})^T \in \mathcal{Q}_j^{\mathbf{t}}$ and $\mathbf{F}_r \neq \mathbf{0}$. In the sequel, these expansions will be expressed as $\mathbf{F} = \mathbf{F}_r + \text{q-h.h.o.t.}$ (“q-h.h.o.t.” stands higher order quasi-homogeneous terms than r). Analogously, if f is a formal function, f can be written as $f = f_k + \text{q-h.h.o.t.}$, being f_k the lowest-degree quasi-homogeneous term of type \mathbf{t} .

If we select the type $\mathbf{t} = (1, 1)$, we are using in fact the Taylor expansion, but in general, each term in the above expansion involves monomials with different degrees. Throughout the paper, we will denote by $\mathbf{D}_0 = (t_1x, t_2y)^T \in \mathcal{Q}_0^{\mathbf{t}}$ (a dissipative quasi-homogeneous vector field) and by $\mathbf{X}_h = (-\partial h/\partial y, \partial h/\partial x)^T$ (Hamiltonian vector field associated to polynomial h). If $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$, then $\mathbf{X}_h \in \mathcal{Q}_r^{\mathbf{t}}$, where $|\mathbf{t}| = t_1 + t_2$. Moreover, it is proved that every $\mathbf{F}_k \in \mathcal{Q}_k^{\mathbf{t}}$ can be expressed as

$$\mathbf{F}_k = \mathbf{X}_h + \mu \mathbf{D}_0 \tag{3}$$

with $h = (\mathbf{D}_0 \wedge \mathbf{F}_k)/(k + |\mathbf{t}|)$ and $\mu = \text{div}(\mathbf{F}_k)/(k + |\mathbf{t}|)$, where $\mathbf{D}_0 \wedge \mathbf{F}_k \in \mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}}$ is the product wedge of both vector fields and $\text{div}(\mathbf{F}_k) \in \mathcal{P}_k^{\mathbf{t}}$ is the divergence of \mathbf{F}_k , see [2]. This sum is known as the conservative-dissipative splitting of a quasi-homogeneous vector field.

In this paper, we deal with nilpotent systems whose quasi-homogeneous expansion respect to the type $\mathbf{t} = (2, n + 1)$ can be written as

$$(\dot{x}, \dot{y})^T = (y, \sigma x^n)^T + \text{q-h.h.o.t.}, \tag{4}$$

with $\sigma = \pm 1$, i.e. nilpotent systems which can be considered as perturbations of a Hamiltonian system. In fact, the nilpotent Hamiltonian system $(\dot{x}, \dot{y})^T = (y, \sigma x^n)^T$ can be written as $(\dot{x}, \dot{y})^T = \mathbf{F}_{n-1}$, with $\mathbf{F}_{n-1} = \mathbf{X}_h \in \mathcal{Q}_{n-1}^{\mathbf{t}}$ being $h = \frac{1}{2(n+1)}(2\sigma x^{n+1} - (n + 1)y^2) \in \mathcal{P}_{2n+2}^{\mathbf{t}}$. We emphasize that the factorization of Hamiltonian associated h on $\mathbb{C}[[x, y]]$ (ring of polynomials in x, y on \mathbb{C}) only has simple factors.

We give our main result, by characterizing the systems (4) which admit an AIIF.

Theorem 2 *System (4) has an AIIF (algebraic inverse integrating factor over $\mathbb{C}((x, y))$) if and only if it is formally orbital equivalent to*

$$(\dot{x}, \dot{y})^T = (y, \sigma x^n)^T + \alpha_M^{(L)} x^M h^L f(h) \mathbf{D}_0, \tag{5}$$

with $h = 2\sigma x^{n+1} - (n + 1)y^2$, $\mathbf{D}_0 = (2x, (n + 1)y)^T$, $\alpha_M^{(L)}$ a real number, f a function with $f(0) = 1$, L a non-negative integer, and $M \in \{0, 1, \dots, n - 1\}$ if $L > 0$ or $M \in \{\lfloor (n + 1)/2 \rfloor, \dots, n - 1\}$ if $L = 0$.

Moreover, if $\alpha_M^{(L)} \neq 0$, then the system (4) is not formally integrable, and if it admits an AIIF, the AIIF is $(h + q\text{-h.h.o.t.})^{\frac{2M+n+3}{2(n+1)}+L}$, up to a multiplicative constant. Otherwise, if $\alpha_M^{(L)} = 0$, system (4) is formally integrable.

Remark 1 If n is even, system (4) has a formal inverse integrating factor if and only if $\alpha_M^{(L)} = 0$, since otherwise the number $\frac{2M+n+3}{2(n+1)}$ is non-integer and hence the inverse integrating factor is not formal.

For n odd, $n = 2m - 1$, the number $\frac{2M+n+3}{2(n+1)}$ is natural if $M = (2k - 1)m - 1$ with k natural. Imposing that $M \leq 2m - 2$, it has that $k = 1$ and $M = m - 1$. So, the system (4) has a formal inverse integrating factor if and only if it is formally orbital equivalent to

$$(\dot{x}, \dot{y})^T = (y, \sigma x^n)^T + x^{(n-1)/2} f(h)(x, ny)^T,$$

with $h = 2\sigma x^{n+1} - (n + 1)y^2$ and f a scalar function.

The following theorem provides a new necessary condition so that an AIIF for the non-degenerate cusp case of nilpotent systems exists.

Theorem 3 Consider a system (2) with $\alpha_2 \neq 0$ (we assume without loss of generality that $\alpha_2 = 1$) and $\beta_2 \neq 0$. If system (2) has an AIIF, then the coefficients $\alpha_3, \alpha_4, \beta_2, \beta_3$ and β_4 satisfy

$$\beta_2(72\beta_2^4 + 2058\alpha_3\beta_2^2 - 10976\alpha_4 + 12005\alpha_3^2) - 2744(5\alpha_3 + \beta_2^2)\beta_3 + 13720\beta_4 = 0,$$

and the AIIF is $(2x^3 - 3y^2 + q\text{-h.h.o.t.})^{7/6}$, up to an multiplicative constant.

From Theorem 3, an AIIF of system (2) is of the form

$$c(2x^3 - 3y^2 + q\text{-h.h.o.t.})^{7/6} = (y^2 - 2/3x^3 + q\text{-h.h.o.t.})^{7/6}e^u,$$

with c a constant and $u = \lg(c(-3)^{7/6})$, that is, Theorem 3 gives a expression of an AIIF more explicit than it given by Theorem 1.

The following theorem provides some necessary conditions which ensure the existence of an AIIF for the nilpotent systems with $\alpha_2 \neq 0$ and $\beta_2 = 0$.

Theorem 4 Consider a system (2) with $\alpha_2 \neq 0$ (we assume without loss of generality that $\alpha_2 = 1$) and $\beta_2 = 0$. The following statements are satisfied:

1. For $\beta_4 \neq \alpha_3\beta_3$, if system (2) has an AIIF then

$$4\beta_5 = \beta_3(4\alpha_4 - 5\alpha_3^2) + 5\alpha_3\beta_4$$

and the AIIF is $(2x^3 - 3y^2 + q\text{-h.h.o.t.})^{11/6}$, up to a multiplicative constant.

2. For $\beta_4 = \alpha_3\beta_3, \beta_5 \neq \alpha_4\beta_3$, if system (2) has an AIIF then

$$80\beta_7 = 140\alpha_3\beta_6 + 7(16\alpha_4 - 25\alpha_3^2)\beta_5 + (175\alpha_4\alpha_3^2 - 140\alpha_3\alpha_5 - 112\alpha_4^2 + 80\alpha_6)\beta_3$$

and the AIIF is $(2x^3 - 3y^2 + q\text{-h.h.o.t.})^{13/6}$, up to a multiplicative constant.

As an application of our results, we study the systems

$$(\dot{x}, \dot{y})^T = (y + a_1 x^{m+1}, x^{2m} + b_1 x^m y)^T, \tag{6}$$

with a_1, b_1 real numbers and $m \geq 1$. Next, we determine the systems (6) with an AIIF.

Theorem 5 *System (6) has an AIIF if and only if either $a_1 = b_1 = 0$ or it satisfies one of the following conditions:*

- (a) $2b_1 = (2m + 1)a_1$,
- (b) $(2m + 1)b_1 = 2(m + 1)^2 a_1$,
- (c) $b_1 = -(m + 1)a_1$.

Moreover, the system (6) is analytically integrable if and only if $b_1 = -(m + 1)a_1$.

The rest of the paper is organized in two sections. In Sect. 2, we give some properties of inverse integrating factor related to its expression and we provide necessary and sufficient conditions for the existence of it. Section 3 contains the proof of Theorems 2–5.

2 Some Properties of an Inverse Integrating Factor

We begin our study by giving an expression of an AIIF of a system (1).

Proposition 6 *If system (1) has an AIIF, then it also admits an inverse integrating factor of the form $V = (W_1/W_2)^d$ with W_1 and W_2 formal series and d a positive rational number.*

Proof From [16, Propositions 1 and 2], specialized to our context, if system (1) has an AIIF, then it also admits an inverse integrating factor V of the specific form $V = \varphi_1^{d_1} \cdots \varphi_s^{d_s}$, with $\varphi_i \in \mathbb{C}[[x, y]]$, non-invertible, irreducible invariants and rational numbers d_i non-zero. (The possibility $s = 0$ is included, with inverse integrating factor 1). So, if we write $d_i = m_i/n_i$ and denote $N = \text{lcm}(\{|n_1|, \dots, |n_s|\})$ and $M = \text{gcd}(\{N|\frac{m_1}{n_1}|, \dots, N|\frac{m_s}{n_s}|\})$, then $V = (\prod_{i=1}^s \varphi_i^{k_i})^{M/N}$, with k_1, \dots, k_s integer numbers non-zero. \square

The two following results provide some properties of inverse integrating factors which are powers of quotient of formal series.

Lemma 7 *We assume that $V = (W_1/W_2)^d$ is an AIIF of $\mathbf{F} = \mathbf{F}_r + q$ -h.h.o.t., with $W_1 = W_{1,m} + q$ -h.h.o.t. and $W_2 = W_{2,n} + q$ -h.h.o.t. where $W_{1,m} \in \mathcal{P}_m^t$ and $W_{2,n} \in \mathcal{P}_n^t$, and d a positive rational number. Then, $(W_{1,m}/W_{2,n})^d$ is an AIIF of \mathbf{F}_r .*

In the particular case $\mathbf{F}_r = \mathbf{X}_h$ with $h \in \mathcal{P}_{r+|t|}^t$, it has that $W_{1,m}/W_{2,n}$ is a rational first integral of \mathbf{X}_h . Moreover, if the factorization of $h \in \mathcal{P}_{r+|t|}^t$ on $\mathbb{C}[x, y]$ only has simple factors then there exists an integer number non-zero k such that $W_{1,m}/W_{2,n} = h^k$.

Proof If V is an AIIF of \mathbf{F} , V satisfies equation $L_{\mathbf{F}}V - V \text{div}(\mathbf{F}) = 0$. From Proposition 6, if we replace V by $(W_1/W_2)^d$ then

$$(W_1/W_2)^{d-1} W_2^{-2} [dW_2 L_{\mathbf{F}}W_1 - dW_1 L_{\mathbf{F}}W_2 - W_1 W_2 \text{div}(\mathbf{F})] = 0.$$

In particular, the lowest degree quasi-homogeneous term of the expression between brackets is also null, i.e.

$$dW_{2,n}L_{\mathbf{F}_r}W_{1,m} - dW_{1,m}L_{\mathbf{F}_r}W_{2,n} - W_{1,m}W_{2,n}\operatorname{div}(\mathbf{F}_r) = 0. \tag{7}$$

Multiplying by $(W_{1,m}/W_{2,n})^{d-1}W_{2,n}^{-2}$ it follows easily that $(W_{1,m}/W_{2,n})^d$ verifies

$$L_{\mathbf{F}_r}(W_{1,m}/W_{2,n})^d - (W_{1,m}/W_{2,n})^d\operatorname{div}(\mathbf{F}_r) = 0. \tag{8}$$

Hence, it is an inverse integrating factor of \mathbf{F}_r , which is algebraic over $\mathbb{C}((x, y))$. We prove the second part. Obviously, if $\mathbf{F}_r = \mathbf{X}_h$, then $\operatorname{div}(\mathbf{F}_r) = 0$ and by (8), $W_{1,m}/W_{2,n}$ is a rational first integral of \mathbf{F}_r .

We see that it is power of the polynomial h , when h only has simple factors in its factorization on $\mathbb{C}[[x, y]]$. We note that the quotient $W_{1,m}/W_{2,n}$ can not be irreducible. In such case, there exist two quasi-homogeneous coprime polynomials $W_{1,\tilde{m}}^*$ and $W_{2,\tilde{n}}^*$ such that $W_{1,m}/W_{2,n} = W_{1,\tilde{m}}^*/W_{2,\tilde{n}}^*$. Since $W_{1,\tilde{m}}^*/W_{2,\tilde{n}}^*$ is a rational first integral of \mathbf{X}_h , then $L_{\mathbf{X}_h}W_{1,\tilde{m}}^*/W_{2,\tilde{n}}^* = 0$, that is,

$$\left(\nabla W_{1,\tilde{m}}^* \cdot \mathbf{X}_h\right) W_{2,\tilde{n}}^* = \left(\nabla W_{2,\tilde{n}}^* \cdot \mathbf{X}_h\right) W_{1,\tilde{m}}^*.$$

Consequently, as $W_{1,\tilde{m}}^*$ and $W_{2,\tilde{n}}^*$ are coprime, there exists $K \in \mathbb{C}[x, y]$, a quasi-homogeneous polynomial, such that $\nabla W_{1,\tilde{m}}^* \cdot \mathbf{X}_h = K W_{1,\tilde{m}}^*$ and $\nabla W_{2,\tilde{n}}^* \cdot \mathbf{X}_h = K W_{2,\tilde{n}}^*$, i.e. $W_{1,\tilde{m}}^*$ and $W_{2,\tilde{n}}^*$ are algebraic invariant curves of \mathbf{X}_h which arrives at the origin. So, if $h = f_1 \cdots f_s$, with f_i irreducible factors on $\mathbb{C}[x, y]$, the unique invariant irreducible curves of \mathbf{X}_h that arrives at the origin are $f_1 = 0, \dots, f_s = 0$. Therefore, $W_{1,\tilde{m}}^* = f_1^{n_1} \cdots f_s^{n_s}$ and $W_{2,\tilde{n}}^* = f_1^{m_1} \cdots f_s^{m_s}$, that is $W_{1,\tilde{m}}^*/W_{2,\tilde{n}}^* = f_1^{k_1} \cdots f_s^{k_s}$ with k_i integer numbers. So, if $M = \operatorname{lcm}\{|k_i|, k_i < 0, i = 1, \dots, s\}$, the function $W_{1,\tilde{m}}^*/W_{2,\tilde{n}}^* h^M$ is a quasi-homogeneous first integral of the system $\dot{\mathbf{x}} = \mathbf{X}_h$ since it is a product of two first integrals. As h only has simple factors in its factorization on $\mathbb{C}[x, y]$, the quasi-homogeneous first integrals of \mathbf{X}_h are h^l , with l a natural number. Therefore, $W_{1,m}/W_{2,n} = W_{1,\tilde{m}}^*/W_{2,\tilde{n}}^* = h^k$ with k integer number non-zero. \square

Lemma 8 Assume that $V_1 = (W_1/W_2)^{\frac{p_1}{q_1}}$ and $V_2 = (\tilde{W}_1/\tilde{W}_2)^{\frac{p_2}{q_2}}$ are AIFs of \mathbf{F} , with W_1, W_2, \tilde{W}_1 and \tilde{W}_2 formal series and $\frac{p_1}{q_1}, \frac{p_2}{q_2}$ rational numbers. Then there exists a natural number l such that $(V_1/V_2)^l$ is a first integral of \mathbf{F} and belongs to $\mathbb{C}((x, y))$.

Proof Taking $l = \operatorname{lcm}\{|q_1|, |q_2|\}$, then $(V_1/V_2)^l = \frac{W_1^{lp_1/q_1} \tilde{W}_2^{lp_2/q_2}}{W_2^{lp_1/q_1} \tilde{W}_1^{lp_2/q_2}}$, i.e. it is a quotient of formal series.

To prove that $(V_1/V_2)^l$ is a integral first of \mathbf{F} , it is enough to prove that V_1/V_2 is a first integral. Indeed,

$$L_{\mathbf{F}} \frac{V_1}{V_2} = \frac{1}{V_2} L_{\mathbf{F}} V_1 - \frac{V_1}{V_2^2} L_{\mathbf{F}} V_2 = \frac{1}{V_2} V_1 \operatorname{div}(\mathbf{F}) - \frac{V_1}{V_2^2} V_2 \operatorname{div}(\mathbf{F}) = 0.$$

Consequently, V_1/V_2 is a first integral of \mathbf{F} . \square
 The following result is key in our study.

Proposition 9 *Let system (1) be with $\mathbf{F} = \mathbf{X}_h + q$ -h.h.o.t. and $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$, where the factorization of h on $\mathbb{C}[x, y]$ only has simple factors. We assume that the system (1) has an AIF.*

Then, system (1) admits an AIF of the form $V = W^q$ being W a formal series $W = h + q$ -h.h.o.t. and q a positive rational number.

Moreover, if system (1) is not formally integrable, then the AIF is unique, up to a multiplicative constant.

Proof Give a number non-zero λ such that the quasi-homogeneous polynomial $H(x, y) = x^{2t_2} + \lambda y^{2t_1} \in \mathcal{P}_{2t_1 t_2}^{\mathbf{t}}$ is not factor of h , we consider the unique solution $(Cs(\theta), Sn(\theta))$ of the initial value problem

$$\frac{d\mathbf{x}}{d\theta} = \mathbf{X}_H(\mathbf{x}), \quad \mathbf{x}(0) = (1, 0)^T.$$

These functions, $Cs(\theta)$ and $Sn(\theta)$, named *generalized trigonometric functions*, are periodic and T will denote their minimal period. For more details, see Dumortier [9]. System (1) by means of the change

$$x = u^{t_1} Cs(\theta), \quad y = u^{t_2} Sn(\theta), \tag{9}$$

with $u \geq 0$ and $\theta \in [0, T)$, and rescaling the time by $dt = (2t_1 t_2 / u^r) d\tau$, is transformed into

$$u' = \frac{du}{d\tau} = -h'(\theta)u + O(u^2), \quad \theta' = \frac{d\theta}{d\tau} = (r + |\mathbf{t}|)h(\theta) + O(u). \tag{10}$$

where $h(\theta) := h(Cs(\theta), Sn(\theta))$.

The equilibria of (10) on $u = 0$ are $(u, \theta) = (0, \theta_j)$, $j = 1 \dots s$, where θ_j are all roots of $h(\theta)$ since we have chosen λ such that H is not a factor of h (otherwise, this factor would not be in the expression of the system (10) since $H(Cs(\theta), Sn(\theta)) = 1$).

The linearization of the system (10) about the fixed points has eigenvalues non-zero (since the factors of h are simple) with different sign; thus, applying a result of Seidenberg [17], we can ensure the existence of an unique solution different from $u = 0$ of the form $\theta - \theta_j + \phi^{(j)}(u, \theta) = 0$, with $\phi^{(j)}(u, \theta) = O(|u, \theta - \theta_j|^2)$. We note that such solution curves are invariant curves of the system (10).

From Proposition 6, if system (1) has an AIF, then it admits an inverse integrating factor of the form $V = (W_1/W_2)^d$. The irreducible factors of W_1 and of W_2 are invariant curves which arrive at the origin, since they are non-invertible, see proof of Proposition 6. So, it has that

$$\frac{W_1(u^{t_1} Cs(\theta), u^{t_2} Sn(\theta))}{W_2(u^{t_1} Cs(\theta), u^{t_2} Sn(\theta))} = \psi(u, \theta) u^m \prod_{j=1}^s \left(\theta - \theta_j + \phi^{(j)}(u, \theta) \right)^{n_j},$$

with m and n_j integer numbers and $\psi(0, \theta_j) \neq 0$ for each $j = 1, \dots, s$. Undoing the change (9), we obtain that

$$\frac{W_1(x, y)}{W_2(x, y)} = \psi(x, y) \prod_{j=1}^s [f_j(x, y) + q\text{-h.h.o.t.}]^{n_j},$$

with $\psi(0, 0) \neq 0$. Moreover, we can assume that $\psi(0, 0) > 0$.

From Lemma 7, we have that $\prod_{j=1}^s (f_j)^{n_j}$ is an AIF of \mathbf{X}_h . In fact, it is a rational first integral of \mathbf{X}_h . As h only has simple factors in $\mathbb{C}[x, y]$, the first integrals, which are quotient of quasi-homogeneous polynomials, are h^n with n a integer number non-zero. If we write $\psi(x, y) = \Psi(x, y)^n$ (it holds $\Psi(0, 0) \neq 0$ and it can be expanded as a series of quasi-homogeneous terms), it has that $W_1/W_2 = (\Psi(h + \phi))^n$, with $h + \phi$ unique. Thus, $V = (h + \phi\text{-h.h.o.t.})^q$ with $q = dn$.

To prove the second part, we see that if there were two distinct AIFs, then the system would be formally integrable. Indeed, let $V_1 = (\Psi_1(h + \phi))^{q_1}$, $V_2 = (\Psi_2(h + \phi))^{q_2}$, $q_1, q_2 \in \mathbb{Q} \setminus \{0\}$, $\Psi_1(0, 0)\Psi_2(0, 0) \neq 0$, two AIFs, we can suppose that $q_2 \geq q_1$. From Lemma 8, there is a natural number l such that $(V_2/V_1)^l = (\Psi_2^{lq_2})/(\Psi_1^{lq_1})(h + \phi)^{l(q_2 - q_1)}$ is a first integral of \mathbf{F} , and is formal since $q_2 \geq q_1$ and Ψ_1, Ψ_2 are invertible series. □

Next, we relate terms of the expression of the formal part of an AIF.

Proposition 10 *Let system (1) with $\mathbf{F} = \mathbf{X}_h + \mu\mathbf{D}_0$, $\mu = \sum_{j \geq 0} \mu_{r+j}$, $\mu_j \in \mathcal{P}_j^t$, for all $j \geq r$. Then, if W^q is an AIF of the system (1), with q an rational number non-zero and $W = \sum_{j \geq 0} W_j \in \mathbb{C}[[x, y]]$, $W_j \in \mathcal{P}_{r+|t|+j}^t$, $W_0 = h$, for each positive integer k it holds:*

$$L_{\mathbf{X}_h} W_k = - \sum_{l=0}^k \left[\frac{r+|t|+l}{q} - (r + |t| + k - l) \right] W_{k-l} \mu_{r+l}. \tag{11}$$

Proof Let denote $\tilde{\mu} = \text{div}(\mu\mathbf{D}_0) = \sum_{j \geq 0} (r + j + |t|)\mu_{r+j}$ and $\tilde{W} = L_{\mathbf{D}_0} W = \sum_{j \geq 0} (r + |t| + j)W_j$. So, if W^q is an AIF of the system (1), then, by definition, $L_{\mathbf{F}} W^q - W^q \text{div}(\mathbf{F}) = qW^{q-1} \left[L_{\mathbf{F}} W - \frac{1}{q} W \text{div}(\mathbf{F}) \right] = 0$, thus

$$L_{\mathbf{X}_h + \mu\mathbf{D}_0} W - \frac{1}{q} W \text{div}(\mathbf{X}_h + \mu\mathbf{D}_0) = L_{\mathbf{X}_h} W + \mu L_{\mathbf{D}_0} W - \frac{1}{q} W \text{div}(\mu\mathbf{D}_0) = 0,$$

i.e. $L_{\mathbf{X}_h} W = \frac{1}{q} W \tilde{\mu} - \mu \tilde{W}$ with $L_{\mathbf{X}_h} W = \sum_{k=0}^{\infty} L_{\mathbf{X}_h} W_k$ and

$$\begin{aligned} & \frac{1}{q} W \tilde{\mu} - \mu \tilde{W} \\ &= \frac{1}{q} \left(\sum_{j \geq 0} W_j \right) \left(\sum_{j \geq 0} (r + j + |t|)\mu_{r+j} \right) - \left(\sum_{j \geq 0} \mu_{r+j} \right) \left(\sum_{j \geq 0} (r + |t| + j)W_j \right) \\ &= \frac{1}{q} \sum_{k=0}^{\infty} \sum_{l=0}^k (r + |t| + l)W_{k-l} \mu_{r+l} - \sum_{k=0}^{\infty} \sum_{l=0}^k (r + |t| + k - l)W_{k-l} \mu_{r+l} \\ &= - \sum_{k=0}^{\infty} \sum_{l=0}^k \left[\frac{r+|t|+l}{q} - (r + |t| + k - l) \right] W_{k-l} \mu_{r+l}, \end{aligned}$$

which establishes the formula (11). □

Now, we provide a series of properties of the inverse integrating factor of the system (1) in order to give conditions that ensure its existence.

For each $k > 0$, we define the linear operator

$$\begin{aligned} \ell_k : \mathcal{P}_k^{\mathbf{t}} &\longrightarrow \mathcal{P}_{k+r}^{\mathbf{t}} \\ \mu_k &\longrightarrow L_{\mathbf{F}_r} \mu_k, \end{aligned}$$

i.e. the Lie derivative of the lowest degree quasi-homogeneous term of \mathbf{F} , and denote by $\text{Cor}(\ell_k)$ to a complementary subspace to the range of the linear operator ℓ_k .

Lemma 11 *Let system (1) be with $\mathbf{F}_r = \mathbf{X}_h \in \mathcal{Q}_r^{\mathbf{t}}$ where $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ only has simple factors in its factorization on $\mathbb{C}[x, y]$.*

If $\mathcal{P}_k^{\mathbf{t}} \neq \{0\}$, there exists a complementary subspace to the range of the operator $\ell_{k+r+|\mathbf{t}|}$ such that $h\text{Cor}(\ell_k) \subset \text{Cor}(\ell_{k+r+|\mathbf{t}|})$.

Proof We suppose that the stated assertion were false. Then, there exist $\lambda_{k+r} \in \text{Cor}(\ell_k) \setminus \{0\}$ and $\nu_{k+r+|\mathbf{t}|} \in \mathcal{P}_{k+r+|\mathbf{t}|}^{\mathbf{t}} \setminus \{0\}$ such that $h\lambda_{k+r} = \ell_{k+r+|\mathbf{t}|}(\nu_{k+r+|\mathbf{t}|})$. This fact implies that

$$h\lambda_{k+r} = \nabla \nu_{k+r+|\mathbf{t}|} \cdot \mathbf{X}_h = -\nabla h \cdot \mathbf{X}_{\nu_{k+r+|\mathbf{t}|}},$$

that is, $h = 0$ is an invariant curve of $\mathbf{X}_{\nu_{k+r+|\mathbf{t}|}}$. Therefore if f is an irreducible factor of h , the curve $f = 0$ is an invariant curve of $\mathbf{X}_{\nu_{k+r+|\mathbf{t}|}}$. As h only has simple factors in its factorization on $\mathbb{C}[x, y]$, it deduces that $\nu_{k+r+|\mathbf{t}|} = h\nu_k$ with $\nu_k \in \mathcal{P}_k^{\mathbf{t}} \setminus \{0\}$. Consequently,

$$h\lambda_{k+r} = \nabla \nu_{k+r+|\mathbf{t}|} \cdot \mathbf{X}_h = \nabla(h\nu_k) \cdot \mathbf{X}_h = h\ell_k(\nu_k),$$

i.e. $\lambda_{k+r} \in \text{Range}(\ell_k)$. This fact contradicts the above assumption. □

Next, we show a class of systems (1) having an AIIF.

Proposition 12 *Let system (1) be, with $\mathbf{F} = \mathbf{X}_h + \lambda g(h)\mathbf{D}_0$, $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$, $\lambda \in \mathcal{P}_{r+k}^{\mathbf{t}}$, $\lambda \neq 0$, $k \geq 0$ and g scalar function with $g(0) = 1$. The function $V(h) = h^{\frac{r+k+|\mathbf{t}|}{r+|\mathbf{t}|}} g(h)$ is an AIIF of (1).*

Proof Applying Euler theorem for quasi-homogeneous function, i.e. $L_{\mathbf{D}_0} f = sf$ with $f \in \mathcal{P}_s^{\mathbf{t}}$, then

$$\begin{aligned} L_{\mathbf{F}} V &= V'(h)L_{\mathbf{F}} h = (r + |\mathbf{t}|)\lambda h g(h) V'(h), \\ &= [(r + k + |\mathbf{t}|)\lambda g(h) + (r + |\mathbf{t}|)\lambda h g'(h)] h^{\frac{r+k+|\mathbf{t}|}{r+|\mathbf{t}|}} g(h) \end{aligned}$$

and

$$\begin{aligned} \text{div}(\mathbf{F}) &= \text{div}(\lambda g(h)\mathbf{D}_0) \\ &= g(h)L_{\mathbf{D}_0} \lambda + \lambda g'(h)L_{\mathbf{D}_0} h + |\mathbf{t}|\lambda g(h) \\ &= (r + k + |\mathbf{t}|)\lambda g(h) + (r + |\mathbf{t}|)\lambda g'(h)h. \end{aligned}$$

So, $L_{\mathbf{F}}V - V\operatorname{div}(\mathbf{F}) = 0$. This completes the proof. □

The following result will be used to prove the sufficiency of Theorem 2.

Proposition 13 *Let system (1) with $\mathbf{F} = \mathbf{X}_h + \mu\mathbf{D}_0$, where the factorization of $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ on $\mathbb{C}[x, y]$ only has simple factors and $\mu = \sum_{j \geq N} \mu_{r+j}$ with $\mu_{r+j} \in \operatorname{Cor}(\ell_j)$, for all $j \geq N > 0$ and $\mu_{r+N} \neq 0$. If V is an AIIF of \mathbf{F} , then $V = h^{\frac{r+N+|\mathbf{t}|}{r+|\mathbf{t}|}} g(h)$, with g a scalar function and $g(0) = 1$, is the unique AIIF of \mathbf{F} up to a multiplicative constant.*

Furthermore, if $\mu = \lambda f(h) + \nu$ with $\lambda \in \operatorname{Cor}(\ell_N) \setminus \{0\}$, f a scalar function, $f(0) = 1$, and $\nu = \sum_{j > N} \nu_{r+j}$, $\nu_{r+j} \in \operatorname{Cor}(\ell_j)$, $\nu_{r+N+l(r+|\mathbf{t}|)} \equiv 0$ for all non-negative integer l , then under these conditions, the system (1) has an AIIF if and only if $\nu \equiv 0$.

Proof From Proposition 9, $V = W^q$ with $W = h + q$ -h.h.o.t. and q a positive rational number.

We prove by induction that W is a function of h . Indeed, let $W = \sum_{j \geq 0} W_j \in \mathbb{C}[[x, y]]$, $W_j \in \mathcal{P}_{r+|\mathbf{t}|+j}^{\mathbf{t}}$. By hypothesis, $W_0 = h$. We thus assume that $W_j = b_l h^l$ for $0 < j < k - 1$, $j = (l - 1)(r + |\mathbf{t}|)$ and null if j is not multiple of $r + |\mathbf{t}|$. By (11), W_k holds

$$\ell_{r+|\mathbf{t}|+k}(W_k) = - \sum_{l=N}^k \left[\frac{r+|\mathbf{t}|+l}{q} - (r + |\mathbf{t}| + k - l) \right] W_{k-l} \mu_{r+l}.$$

From Lemma 11, $h^n \mu_{r+l} \in \operatorname{Cor}(\ell_{l+n(r+|\mathbf{t}|)})$, $n \in \mathbb{N}$, hence $\ell_{r+|\mathbf{t}|+k}(W_k) \in \operatorname{Cor}(\ell_{r+|\mathbf{t}|+k})$ and so $\ell_{r+|\mathbf{t}|+k}(W_k) = 0$. Therefore, W_k is a quasi-homogeneous polynomial first integral of \mathbf{X}_h , i.e. it is a function of h , in fact, $W_k = b_k h^k$ with $b_k \in \mathbb{R}$. Thus the induction is complete.

We now prove that $q = \frac{r+|\mathbf{t}|+N}{r+|\mathbf{t}|}$. For $k = N$, equation (11) becomes

$$L_{\mathbf{X}_h} W_N = - \left[\frac{r+|\mathbf{t}|+N}{q} - (r + |\mathbf{t}|) \right] h \mu_{r+N}.$$

From Lemma 11, it has that $h \mu_{r+N} \in \operatorname{Cor}(\ell_{N+r+|\mathbf{t}|})$, hence $L_{\mathbf{X}_h} W_N = 0$. Consequently, $q = \frac{r+|\mathbf{t}|+N}{r+|\mathbf{t}|}$ since $h \mu_{r+N} \neq 0$. Summarizing, it has that

$$W^q = \left(h(1 + \sum_{i \geq 1} b_i h^i) \right)^{\frac{r+|\mathbf{t}|+N}{r+|\mathbf{t}|}} = h^{\frac{r+|\mathbf{t}|+N}{r+|\mathbf{t}|}} g(h), \text{ with } g(0) = 1.$$

In this case, the system (1) is in normal form and from Algaba et al. [2, Theorem 3.19], it is not formally integrable. Thus, from Proposition 9, the AIIF is unique up to a multiplicative constant.

We see the second part. Proposition 12 proves the sufficiency.

To prove the necessity, we assume that $\nu_{r+j} \equiv 0$ for $N < j < j_0$ and we must prove that $\nu_{r+j_0} = 0$. We also can certainly assume that $j_0 - N$ is not a multiple of $r + |\mathbf{t}|$ since otherwise $\nu_{r+N+j_0} = 0$.

If \mathbf{F} has an AIIF, we have proved that \mathbf{F} admits an AIIF of the form $W^{\frac{r+|\mathbf{t}|+N}{r+|\mathbf{t}|}}$ with $W = h + q$ -h.h.o.t. $\in \mathbb{C}[[x, y]]$ and $W = W(h)$ with $W_0 = h$.

Taking into account the expressions of v , λ and $f(h)$, the quasi-homogeneous terms of $\lambda f(h) + v$ are given by $(\lambda f(h) + v)_{r+N} = \lambda$, $(\lambda f(h) + v)_{r+l} = \lambda(f(h))_{l-N}$, with $l = N + 1, \dots, j_0 - 1$ and $(\lambda f(h) + v)_{r+j_0} = \lambda(f(h))_{j_0-N} + v_{r+j_0}$. Hence, equation (11) for $k = j_0$, becomes

$$\begin{aligned} L_{X_h} W_{j_0} &= - \sum_{l=N}^{j_0} \left[\frac{(r+|t|+l)(r+|t|)}{r+|t|+N} - (r + |t| + j_0 - l) \right] W_{j_0-l} (\lambda f(h) + v)_{r+l} \\ &= (j_0 - N) W_{j_0-N} \lambda \\ &\quad - \sum_{l=N+1}^{j_0-1} \left[\frac{(r+|t|+l)(r+|t|)}{r+|t|+N} - (r + |t| + j_0 - l) \right] W_{j_0-l} \lambda (f(h))_{l-N} \\ &\quad - \frac{(r+|t|)(j_0-N)}{r+|t|+N} h (\lambda f(h)_{j_0-N} + v_{r+j_0}). \end{aligned}$$

As $j_0 - N$ is not a multiple of $r + |t|$ then $W_{j_0-N} \equiv f(h)_{j_0-N} \equiv 0$ and $W_{j_0-l} (f(h))_{l-N} \equiv 0$ for $N < l < j_0$, hence

$$L_{X_h} W_{j_0} = - \frac{(r+|t|)(j_0-N)}{r+|t|+N} h v_{r+j_0}.$$

As $(r + |t|)(j_0 - N) \neq 0$, then $h v_{r+j_0} \in \text{Range}(\ell_{r+|t|+j_0})$, and from Lemma 11, $h v_{r+j_0} \in \text{Cor}(\ell_{r+|t|+j_0})$. This clearly forces $v_{r+j_0} \equiv 0$. \square

3 Proofs of the Main Results

We state the well-known relationship between inverse integrating factors of formally orbital equivalent vector fields.

Proposition 14 *Let Φ be a diffeomorphism and η a function on $U \subset \mathbf{R}^2$ such that $\det D\Phi$ has no zero on U and $\eta(\mathbf{0}) \neq 0$. If $V(\mathbf{x}) \in \mathbb{C}[[x, y]]$ is an inverse integrating factor of the system (1), then $\eta(\mathbf{y})(\det(D\Phi(\mathbf{y})))^{-1}V(\Phi(\mathbf{y}))$ is an inverse integrating factor of $\dot{\mathbf{y}} = \Phi_*(\eta\mathbf{F})(\mathbf{y}) := D\Phi(\mathbf{y})^{-1}\eta(\mathbf{y})\mathbf{F}(\Phi(\mathbf{y}))$.*

Proof of Theorem 2 We prove that the condition is sufficient. If $\alpha_M^{(L)} = 0$, the polynomial $h(x, y) = (2\sigma x^{n+1} - (n + 1)y^2)^m$, m any natural number, is a polynomial first integral and, in particular, it is an inverse integrating factor. Thus, by Proposition 14, if we perform the transformation which brings this system to the system (4), then system (4) admits an AIF, but it is not unique modulus a multiplicative constant.

In the case, $\alpha_M^{(L)} \neq 0$, system (5) has the form given in Proposition 12 for $h = 2\sigma x^{n+1} - (n + 1)y^2$, $\lambda = \alpha_M^{(L)} x^M h^L$ and $g(h) = f(h)$. Therefore, the system admits the AIF, $V(h) = h^{\frac{2M+n+3}{2(n+1)}+L} f(h)$, up to a multiplicative constant. From Proposition 14, the system (4) has the AIF, $V = (h + q\text{-h.o.t.})^{\frac{2M+n+3}{2(n+1)}+L}$, up to a multiplicative constant.

We now prove that the condition is necessary. From Algaba et al. [3], the system (4) is formally orbital equivalent to

$$(\dot{x}, \dot{y})^T = (y, \sigma x^n)^T + \sum_{j=\lfloor (n+1)/2 \rfloor}^{n-1} \alpha_j^{(0)} x^j \mathbf{D}_0 + \sum_{l=1}^{\infty} \sum_{j=0}^{n-1} \alpha_j^{(l)} x^j h^l \mathbf{D}_0, \tag{12}$$

i.e. there exist a near-identity change of variables $\mathbf{y} = \phi(\mathbf{x})$ and a reparameterization of the time, such that the system (4) is transformed into (12).

If $\alpha_j^{(l)} = 0$, for all j and l then $\alpha_M^{(L)} = 0$, the system is a Hamiltonian system, hence it has a formal inverse integrating factor.

Otherwise, if $L = \min\{l, \alpha_j^{(l)} \neq 0\}$, $M = \min\{j, \alpha_j^{(L)} \neq 0\}$, system (12) is of the form $(\dot{x}, \dot{y})^T = \mathbf{F} = \mathbf{X}_h + (\lambda f(h) + \nu) \mathbf{D}_0$, with $\lambda = \alpha_M^{(L)} x^M h^L$, $N = 2M + 2L(n + 1) - n + 1$, $\nu = \sum_{j > 2M + 2(n+1)L} \nu_j$, $\nu_j \in \text{Cor}(\ell_j)$, $\nu_{2M + 2(n+1)(l+L)} \equiv 0$ for all non-negative integer l , and f a scalar function with $f(0) = 1$. From Proposition 13, the system (12) admits an AIIF if and only if $\nu \equiv 0$, i.e. system (12) agrees with system (5). In this case, as $\alpha_M^{(L)} \neq 0$, from [2, Theorem 3.19], the system (5) is not formally integrable, and from Proposition 13, the unique AIIF, up to a multiplicative constant, is of the form $\tilde{V} = (h + \text{q-h.h.o.t})^{\frac{2M+n+3}{2(n+1)} + L}$. Undoing the change and by using Proposition 14, the proof is complete. \square

The coefficients of the formal normal form (12) of the system (4) have been obtained by using the procedure given in Algaba et al. [1]. This method consists in a recursive procedure to compute quasi-homogeneous normal form under equivalence, which uses the Lie triangle.

Proof of Theorem 3 From Algaba et al. [2], the system (2) with $\alpha_2 = 1$ is orbitally equivalent to

$$(\dot{x}, \dot{y})^T = (y, x^2)^T + \alpha_1^{(0)} x \mathbf{D}_0 + \alpha_0^{(1)} h \mathbf{D}_0 + \text{q-h.h.o.t.} \tag{13}$$

with

$$\begin{aligned} \alpha_1^{(0)} &= \beta_2, \\ \alpha_0^{(1)} &= -\beta_2(-72\beta_2^4 - 2058\alpha_3\beta_2^2 + 10976\alpha_4 - 12005\alpha_3^2) \\ &\quad - 2744(5\alpha_3 + \beta_2^2)\beta_3 + 13720\beta_4. \end{aligned}$$

If the system (2) has an AIIF then, from Theorem 2, $\alpha_0^{(1)} = 0$ since $\alpha_1^{(0)} = \beta_2 \neq 0$. In this case, the system (13) is a normal form up to order 6, i.e. it agrees with the system (5) up to order 6 for $M = 1$, $L = 0$ and $n = 2$, so the exponent given in Theorem 2 is $\frac{2M+n+3}{2(n+1)} + L = 7/6$. \square

Proof of Theorem 4 From Algaba et al. [2], the system (2), with $\alpha_2 = 1$ and $\beta_2 = 0$, is orbitally equivalent to

$$(\dot{x}, \dot{y})^T = (y, x^2)^T + \alpha_0^{(1)} h \mathbf{D}_0 + \alpha_1^{(1)} x h \mathbf{D}_0 + \alpha_0^{(2)} h^2 \mathbf{D}_0 + \text{q-h.h.o.t.} \tag{14}$$

whose first two coefficients are, up to a multiplicative constants non-zero,

$$\begin{aligned} \alpha_0^{(1)} &= -\alpha_3\beta_3 + \beta_4, \\ \alpha_1^{(1)} &= 4\beta_5 + (-4\alpha_4 + 5\alpha_3^2)\beta_3 - 5\alpha_3\beta_4. \end{aligned}$$

We assume that system (2) has an AIIF:

If $\alpha_0^{(1)} \neq 0$, that is $\beta_4 \neq \alpha_3\beta_3$, from Theorem 2, it has that $\alpha_1^{(1)} = 0$, i.e.

$$4\beta_5 = \beta_3(4\alpha_4 - 5\alpha_3^2) + 5\alpha_3\beta_4.$$

In this case, the system (14) is a normal form up to order 8, i.e. it agrees with the system (5) up to order 8 for $M = 0, L = 1$ and $n = 2$, so the exponent given in Theorem 2 is $\frac{2M+n+3}{2(n+1)} + L = 11/6$.

If $\alpha_0^{(1)} = 0$, that is $\beta_4 = \alpha_3\beta_3$, the coefficients $\alpha_1^{(1)}$ and $\alpha_0^{(2)}$ are, up to a multiplicative constants non-zero,

$$\begin{aligned} \alpha_1^{(1)} &= 4\beta_5 - \alpha_4\beta_3, \\ \alpha_0^{(2)} &= -140\alpha_3\beta_6 - (175\alpha_3^2\alpha_4 + 80\alpha_6 - 140\alpha_3\alpha_5 - 112\alpha_4^2)\beta_3 \\ &\quad - 7(16\alpha_4 - 25\alpha_3^2)\beta_5 + 80\beta_7. \end{aligned}$$

From Theorem 2, it has that $\alpha_0^{(2)} = 0$.

In this case, the system (14) is a normal form up to order 16, i.e. it agrees with the system (5) up to order 16 for $M = 1, L = 1$ and $n = 2$, so the exponent given in Theorem 2 is $\frac{2M+n+3}{2(n+1)} + L = 13/6$. This finishes the proof. \square

Proof of Theorem 5 For $a_1 = b_1 = 0$, the system has an AIIF since it is a Hamiltonian system. In the case $a_1^2 + b_1^2 \neq 0$, we will do our study depending on the first coefficient non-zero of (12) with $n = 2m, \sigma = 1$. For $m \geq 1$, the first coefficient is $\alpha_m^{(0)} = b_1 + (m + 1)a_1$.

If $b_1 + (m + 1)a_1 = 0$, system (6) is a Hamiltonian system. Hence, it has a polynomial inverse integrating factor. Furthermore, $(2x^{2m+1} - (2m + 1)y^2)^k$, with k natural number, is a polynomial first integral. Otherwise, $b_1 + (m + 1)a_1 \neq 0$, the coefficient of the second lowest-degree term of (12) is

$$\begin{aligned} \alpha_{m+1}^{(0)} &= [2b_1 - (2m + 1)a_1] \left[(2m + 1)b_1 - 2(m + 1)^2a_1 \right] \text{ for } m > 1, \\ \alpha_0^{(1)} &= (2b_1 - 3a_1)(3b_1 - 8a_1)(48a_1^2 - 197b_1a_1 + 12b_1^2), \text{ for } m = 1. \end{aligned}$$

If $2b_1 - (2m + 1)a_1 = 0, m \geq 1$, it is easy to check that $(2x^{2m+1} - (2m + 1)y^2)^{1 + \frac{1}{4m+2}}$ is an AIIF of the system (6).

If $(2m + 1)b_1 - 2(m + 1)^2a_1 = 0$, the system admits the inverse integrating factor $(\frac{1}{2m+1}x^{2m+1} - \frac{1}{2}y^2 + \frac{a_1}{2m+1}x^{m+1}y - \frac{a_1^2}{2(2m+1)^2}x^{2m+2})^{1 + \frac{1}{4m+2}}$.

If $m = 1, b_1 + 2a_1 \neq 0, 2b_1 - 3a_1 \neq 0, 3b_1 - 8a_1 \neq 0$ and $48a_1^2 - 197b_1a_1 + 12b_1^2 = 0$, the third coefficient of the normal form is

$$\alpha_1^{(1)} = 9036b_1^4 - 265812b_1^3a_1 - 4436911b_1^2a_1^2 - 1063248a_1^3b_1 + 144576a_1^4.$$

Computing the resultant of both curves respect to a_1 , we check that a_1 and b_1 must be null. \square

Acknowledgments This work has been partially supported by *Ministerio de Ciencia y Tecnología, Plan Nacional I+D+I* co-financed with FEDER funds, in the frame of the projects MTM2007-64193 and MTM2010-20907-C02-02, and by *Consejería de Educación y Ciencia de la Junta de Andalucía (FQM-276 and P08-FQM-03770)*.

References

1. Algaba, A., Freire, E., Gamero, E., García, C.: An algorithm for computing quasi-homogeneous formal normal forms under equivalence. *Acta Appl. Math.* **80**, 335–359 (2004)
2. Algaba, A., Gamero, E., García, C.: The integrability problem for a class of planar systems. *Nonlinearity* **22**(2), 395–420 (2009)
3. Algaba, A., García, C., Reyes, M.: Existence of an inverse integrating factor, center problem and integrability of a class of nilpotent systems (Submitted for publication)
4. Algaba, A., García, C., Reyes, M.: Local bifurcation of limit cycles and integrability of a class of nilpotent systems of differential equations. *Appl. Math. Comput.* **215**, 314–323 (2009)
5. Chavarriga, J.: Integrable systems in the plane with a center type linear part. *Appl. Math. (Warsaw)* **22**, 285–309 (1994)
6. Chavarriga, J., Giacomini, H., Giné, J., Llibre, J.: On the integrability of two-dimensional flows. *J. Differ. Equ.* **157**, 163–182 (1999)
7. Chavarriga, J., Giacomini, H., Giné, J., Llibre, J.: Darboux integrability and the inverse integrating factor. *J. Differ. Equ.* **194**, 116–139 (2003)
8. Christopher, C., Mardesic, P., Rousseau, C.: Normalizable, integrable, and linealizable saddle points for complex quadratic systems in \mathbb{C}^2 . *J. Dyn. Control Syst.* **9**, 311–363 (2003)
9. Dumortier, F.: Singularities of vector fields on the plane. *J. Differ. Equ.* **23**, 53–106 (1977)
10. Enciso, A., Peralta-Salas, D.: Existence and vanishing set of inverse integrating factors for analytic vector fields. *Bull. Lond. Math. Soc.* **41**, 1112–1124 (2009)
11. Ferragut, A., Llibre, J., Mahdi, A.: Polynomial inverse integrating factors for polynomial vector fields. *Discrete Contin. Dyn. Syst.* **17**, 387–395 (2006)
12. García, I., Giacomini, H., Grau, M.: The inverse integrating factor and the Poincaré map. *Trans. Am. Math. Soc.* **362**(7), 3591–3612 (2010)
13. García, I., Grau, M.: A survey on the inverse integrating factor. *Qual. Theory Dyn. Syst.* **9**(1–2), 115–166 (2010)
14. García, I., Shafer, D.: Integral invariants and limit sets of planar vector fields. *J. Differ. Equ.* **217**(2), 363–376 (2005)
15. Giacomini, H., Llibre, J., Viano, M.: On the nonexistence, existence and uniqueness of limit cycles. *Nonlinearity* **9**, 501–516 (1996)
16. Prelle, M.J., Singer, M.F.: Elementary first integrals of differential equations. *Trans. Amer. Math. Soc.* **279**, 215–229 (1983)
17. Seidenberg, A.: Reduction of singularities of the differential equation $Ady = Bdx$. *Am. J. Math.* **90**, 248–269 (1968)
18. Singer, M.F.: Liouvillian first integrals of differential equations. *Trans. Amer. Math. Soc.* **333**, 673–688 (1992)
19. Walcher, S.: Local integrating factors. *J. Lie Theory* **13**, 279–289 (2003)