Quantum Phase Transitions in Coupled Systems
An application of Catastrophe Theory

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The Lipkin model

The two-fluid Lipkin model

- Algebraic structure
- The Consistent-Q-like Lipkin Hamiltonian

Energy surface: mean field

Phase transition and Quantum Phase Transition in a nutshell

Phase diagram

With a little help from my ... Catastrophe Theory
Lipkin model

The SU(2) algebra

\[ J^+ = t^s, \quad J^- = s^t, \quad J^0 = \frac{1}{2}(t^t - s^s), \quad J'^0 = J^0 + N/2 \]

\[ [J^+, J^-] = 2J^0, \quad [J^0, J^\pm] = \pm J^\pm \]

The Hamiltonian

\[ H = aJ'^0 + b(J^+ + J^-) + c(J^+ J^-) + d((J^+)^2 + (J^-)^2) + e(J^+ J'^0 + J'^0 J^-) + f(J'^0)^2 \]

The Consistent-Q-like Hamiltonian

\[ H = x n_t + \frac{x - 1}{N} Q(y) Q(y), \]

where \( n_t = t^t \) and \( Q(y) = s^t t + t^s + y t^t \)
Two-fluid Lipkin model

The Hamiltonian

\[ H = H_1 + H_2 + H_{12}, \]

where

\[ H_i = a_i J'_i + b_i (J_i^+ + J_i^-) + c_i (J_i^+ J_i^-) + d_i ((J_i^+)^2 + (J_i^-)^2) \]
\[ + e_i (J_i^+ J'_i + J'_i J_i^-) + f_i (J'_i)^2 \]

\[ H_{12} = w_1 (J_1^+ J_2^+ + J_1^- J_2^-) + w_2 (J_1^+ J_2^- + J_1^- J_2^+) + w_3 (J_1^+ J'_2 + J_1^- J'_2) \]
\[ + w_4 (J'_1 J_2^+ + J'_1 J_2^-) + w_5 J'_1 J'_2, \]

being

\[ J'_i = J_i^0 + \frac{N}{2}. \]
Consistent-Q-like two-fluid Lipkin model

\[ H = x (n_{t1} + n_{t2}) - \frac{1 - x}{N_1 + N_2} Q(y_1, y_2) \cdot Q(y_1, y_2) \]

where

\[ n_{ti} = t_i^\dagger t_i \]
\[ Q^{(y_1, y_2)} = (Q_1^{y_1} + Q_2^{y_2}) \]
\[ Q_i^{y_i} = s_i^\dagger t_i + t_i^\dagger s_i + y_i \left( t_i^\dagger t_i \right) \]
Mean field, energy surface

**Ground state**

\[ |g\rangle = \frac{1}{\sqrt{N_1!N_2!}} (\Gamma_1^\dagger)^{N_1} (\Gamma_2^\dagger)^{N_2} |0\rangle \]

\[ \Gamma_i^\dagger = \frac{1}{\sqrt{1 + \beta_i^2}} (s_i^\dagger + \beta_i t_i^\dagger) \]

**Energy**

\[
\frac{E(\beta_1, \beta_2, x, y_1, y_2)}{N_1 + N_2} = \frac{x}{2} \left( \frac{\beta_1^2}{1 + \beta_1^2} + \frac{\beta_2^2}{1 + \beta_2^2} \right) \\
- \frac{1 - x}{4} \left( \frac{1}{(1 + \beta_1^2)^2} (2 \beta_1 + y_1 \beta_1^2)^2 \right) \\
+ \frac{1}{(1 + \beta_2^2)^2} (2 \beta_2 + y_2 \beta_2^2)^2 \\
+ 2 \frac{1}{(1 + \beta_1^2)} \frac{1}{(1 + \beta_2^2)} (2 \beta_1 + y_1 \beta_1^2)(2 \beta_2 + y_2 \beta_2^2) \]
Figure: Contour plot of a selected two-fluid Lipkin Hamiltonian.
Macroscopic/Classical Phase Transitions

Definition of phase and phase transition

- **Phase**: state of matter that is uniform throughout, not only in chemical composition but also in physical properties.
- **Phase Transition**: abrupt change in one or more properties of the system.

The phase of the system

- Most stable phase of matter is the one with the lowest thermodynamic potential $\Phi$. This is a function of some parameters that are allowed to change ($F(T,V), F(T,B); G(T,p), G(T,M)$).
- $\Phi$ is analogous to the potential energy, $V(x)$, of a particle in a one dimensional well. The system looks for the minimum energy going into the bottom of the potential.
Macroscopic/Classical Phase Transitions

- **Control parameter**: variable that affects the system, it can be changed smoothly and “arbitrarily”.
- **Order parameter**: observable that changes as a function of the control parameter and that defines the “phase” of the system.
- **Ordered and disordered phases** correspond to a value of the order parameter equal and different from zero, respectively.
- **Order of a phase transition**: order of the first derivative of the Gibbs potential with respect to the control parameter that first experiences a discontinuity: first, continuous (second order).
Examples of Macroscopic Phase Transitions

First order phase transition.  
Liquid-gas

Second order phase transition.  
Paramagnetic-ferromagnetic
What is happening at the phase transition point?

- First order phase transition
- Second order phase transition
QPT occurs at some critical value, $x_c$, of the control parameter $x$ that controls an interaction strength in the system’s Hamiltonian $H(x)$. It is implicit a zero temperature.

\[ \hat{H} = x \hat{H}_1 + (1 - x) \hat{H}_2 \]

At the critical point:

- The ground state energy is nonanalytic.
- The gap $\Delta$ between the first excited state and the ground state vanishes.
The variation of the order parameter
Phase diagram
Figure: Phase diagram of the Q-like two-fluid Lipkin model
Numerical results 1

**Figure:** $y_1 = 0$, $y_2 = 0$
Figure: $y_1 = 0.5$, $y_2 = -0.5$
Figure: \( y_1 = 1, \ y_2 = 1 \)
Figure: $y_1 = 1, y_2 = -1$
Numerical results 4b

Figure:

\[ y_1 = 1, \quad y_2 = -1 \]
Figure: $y_1 = 1$, $y_2 = -1$
What is for Catastrophe Theory?

Some notes


- Catastrophe theory (CT) is framed in the theory of singularities for differentiable mappings and in the theory of bifurcations, therefore it deals with singularities of smooth real-valued functions and tries to classify such singularities.

- CT attempts to study how the qualitative nature of the solutions of equations depends on the parameters that appear in the equations (Gilmore 1981).

- CT explains how the state of a system can change suddenly under a smooth change in the control variables.
Let us assume a system described by a real family of potentials:

$$V(x, \lambda) \in \mathbb{R}$$

where $x \in \mathbb{R}^n$ are the state (order) variables and $\lambda \in \mathbb{R}^r$ are the control parameters.

In this family one can find three types of points:

- Regular points: $\nabla V \neq 0$.
- Morse points (isolated critical points):
  $\nabla V = 0$ and $|\mathcal{H}_{ij}| \neq 0$.
- Non-Morse points (degenerated critical points):
  $\nabla V = 0$ and $|\mathcal{H}_{ij}| = 0$. 
Relevant theorems

- Morse lemma for isolated critical points.
  \[ V(x) \rightarrow x^2 \]

- Thom theorem for degenerated critical points.
  \[ V(x) \rightarrow g(x) + \text{unfolding} \]

- Splitting lemma for potential with several variables.
  \[ V(x, y, z) \rightarrow g(x) + \text{unfolding} + y^2 - z^2 \]
Substitution of $V(x, \lambda)$ by a truncated Taylor expansion $V(x, \lambda)_{pol}$, being the germ the higher order term (the order of the Taylor expansion is the determinacy...).

Establish the mapping between $V(x, \lambda)_{pol}$ and a canonical form through a nonlinear change of variables.

Work out $V(x, \lambda)_{pol}$ for getting the bifurcation and the Maxwell set.
A local Taylor expansion

- Let us make a Taylor expansion around $\beta_1 = 0$ and $\beta_2 = 0$.
- Moreover we will take as variables the eigenvectors of the Hessian matrix.

$$\mathcal{H} = \begin{pmatrix}
\frac{\partial^2 E}{\partial \beta_1^2} & \frac{\partial^2 E}{\partial \beta_1 \partial \beta_2} \\
\frac{\partial^2 E}{\partial \beta_2 \partial \beta_1} & \frac{\partial^2 E}{\partial \beta_2^2}
\end{pmatrix} = \begin{pmatrix}
3x - 2 & 2x - 2 \\
2x - 2 & 3x - 2
\end{pmatrix}.$$  

The two eigenvalues are $5x - 4$ and $x$, and the corresponding eigenvectors eigenvectors are,

$$\beta_a = \frac{1}{2} (\beta_1 + \beta_2),$$
$$\beta_b = \frac{1}{2} (\beta_1 - \beta_2).$$
A different view of the energy surface

- Let us make a Taylor expansion around $\beta_1 = 0$ and $\beta_2 = 0$.
- Moreover we will take as variables the eigenvectors of the Hessian matrix.

$$E(x, y, y', \beta_a, \beta_b) = \frac{(5x - 4)\beta_a^2 + 4(x - 1)y\beta_a^3 + (8 - 9x + y^2(x - 1))\beta_a^4 + \Theta(\beta_a^5) + x\beta_b^2 + \Theta(\beta_a\beta_b^2, \beta_b\beta_a^2)}{N}$$

with $y = (y_1 + y_2)/2$ and $y' = (y_1 - y_2)/2$.

In order to cancel the higher order ($\beta_a^i\beta_b^j$ with $j > 1$) we have to implement a nonlinear transformation in $\beta_b$

$$\tilde{\beta}_b = \beta_b + \sum_{i+j>1} a_{ij}\beta_a^i\beta_b^j$$

$\beta_a$ becomes the essential variable while $\beta_b$ the non-essential one.
A different view of the energy surface

For \( y' = 0 \)

\[
\frac{E(x, y, \beta_a, \beta_b)}{N} = (5x - 4)\beta_a^2 + 4(x - 1)y\beta_a^3 + \left( y^2(x - 1) + 8 - 9x \right)\beta_a^4 + O(\beta_a^5) + x\beta_b^2
\]
A different view of the energy surface

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\]

For $y = 0$

\[
\frac{E(x, y', \beta_a, \beta_b)}{N} = (5x - 4)\beta_a^2 + \left(8 - 9x - \frac{16(x - 1)^2y'^2}{x}\right)\beta_a^4 \\
+ \frac{1}{2} \left(26x - 24 + \frac{128(x - 1)^3y'^2(y'^2 - 6)}{x^2}\right)\beta_a^6 \\
+ O(\beta_a^7) + x\beta_b^2
\]
For the general Hamiltonian

\[
\frac{E(x, y, y', \beta_a, \beta_b)}{N} = (5x - 4)\beta_a^2 + 4(x - 1)y\beta_a^3 + \left(8 - 9x + y^2(x - 1) + \frac{16y'^2}{x}(2x - x^2 - 1)\right)\beta_a^4
\]

\[
+ \frac{8(x - 1)y \left((6y'^2 - 1)x^2 - 14y'^2x + 8y'^2\right)}{x^2} \beta_a^5
\]

\[
+ \frac{1}{x^3} \left((64y'^4 - 384y'^2 - 2y^2(82y'^2 + 1) + 13)x^4 + 2(-96y'^4 + 576y'^2 \right)
\]

\[
+ y^2(372y'^2 + 1) - 6)x^3 + 4y'^2(48(y'^2 - 6) - 313y^2)x^2
\]

\[
+ 32y'^2(29y^2 - 2y'^2 + 12)x - 256y^2y'^2 \right) \beta_a^6
\]

\[
+ O(\beta_a^7) + x\beta_b^2
\]
A different view of the energy surface

For the general Hamiltonian

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+ \frac{8(x - 1)y \left((6y'^2 - 1)x^2 - 14y'^2x + 8y'^2\right)}{x^2} \beta_a^5 \\
+ \frac{1}{x^3} \left((64y'^4 - 384y'^2 - 2y^2(82y'^2 + 1) + 13)x^4 + 2(-96y'^4 + 576y'^2)\right) \\
+ y^2(372y'^2 + 1) - 6)x^3 + 4y'^2(48(y'^2 - 6) - 313y^2)x^2 \\
+ 32y'^2(29y^2 - 2y'^2 + 12)x - 256y^2y'^2 \beta_a^6 \\
+ O(\beta_a^7) + x\beta_b^2
\]

The function is 6 — determined. The relevant elementary catastrophe of this model will be the butterfly (A_+_5). Note that we only have 3 control parameters, though the universal unfolding of the butterfly has 4 parameters.
Summary and conclusions

- We have presented the Consistent-Q like doble Lipkin Hamiltonian.
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We found that the appropriated catastrophe is the \textit{butterfly} \((A_+5)\).
Summary and conclusions

- We have presented the Consistent-Q like doble Lipkin Hamiltonian.
- We have analyze the appearance of phase transitions in this model.
- We used Catastrophe Theory to disentangle the order of the transitions.
- We found that the appropriated catastrophe is the butterfly ($A_{1+5}$).

Thank you!
Numerical results

Figure: $x = 0.5$, $y_1 = 1$
Definition of \( h(\mathbf{x}, \lambda) = V(\mathbf{x} + \mathbf{x}^0, \lambda + \lambda^0) - V(\mathbf{x}^0, \lambda^0) \), where \((\mathbf{x}^0, \lambda^0)\) correspond to a degenerated critical point.

Definition of the germ: \( g(\mathbf{x}) = h(\mathbf{x}, 0) \).

Calculation of the determinacy and the codimension of \( g(\mathbf{x}) \) through the \( k \)-jet of \( g(\mathbf{x}) \) (truncated Taylor expansion with \( k \) term).

Study the \( k \)-transversality of \( g(\mathbf{x}) \) in order to establish the isomorphism between \( h(\mathbf{x}, \lambda) \) and a canonical unfolding of \( g(\mathbf{x}) \).

Note that it is only possible to prove the existence of an isomorphism but this DOES NOT provides the necessary change of coordinates.
Figure: Energy spectrum (per boson) for a Lipkin model with $N = 50$, $\alpha = x$ and $y = 0$. 
The coordinates

Figure: Coordinates of the Q-like two-fluid Lipkin space.

\[ H = x (n_{t1} + n_{t2}) - \frac{1 - x}{N_1 + N_2} Q(y_1, y_2) \cdot Q(y_1, y_2) \]